Modeling of DC motor

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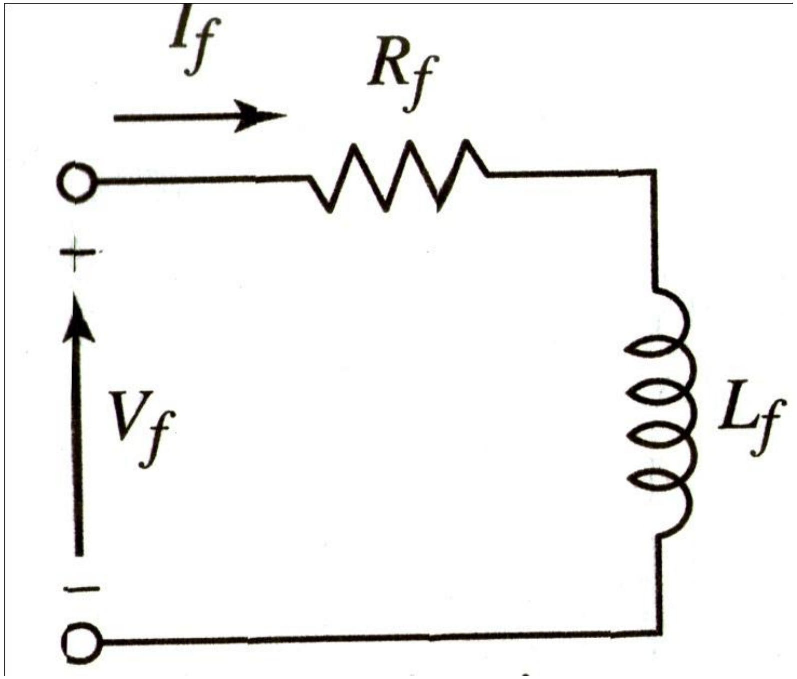
DC Motor Electrical Equations

Considering the figure shown below:



The impedance of a DC motor's armature can be modelled as a resistance *R* in series with the parallel combination of an inductance *L* and a second resistance *RL* .

- However, as the conducting armature begins to rotate in the magnetic field produced by the stator, a voltage called the **back Emf** Eg is induced in the armature windings opposing the applied voltage.

- The back Emf is proportional to the angular velocity of the rotor ω in rad/sec:

$$E_g = K_e \omega \text{ ----------------(1)}$$

- where,

the proportionality constant Ke is called the **voltage constant** of the motor.

Applying Kirchoff's voltage law,

$$Va = La \frac{d}{dt} Ia + Ra\, Ia + Eg$$

$$Va = La \frac{d}{dt} Ia + Ra\, Ia + Ke\, \omega$$

where,
La = inductance of armature,
Ia = armature current,
Va = armature voltage
Ra = armature resistance

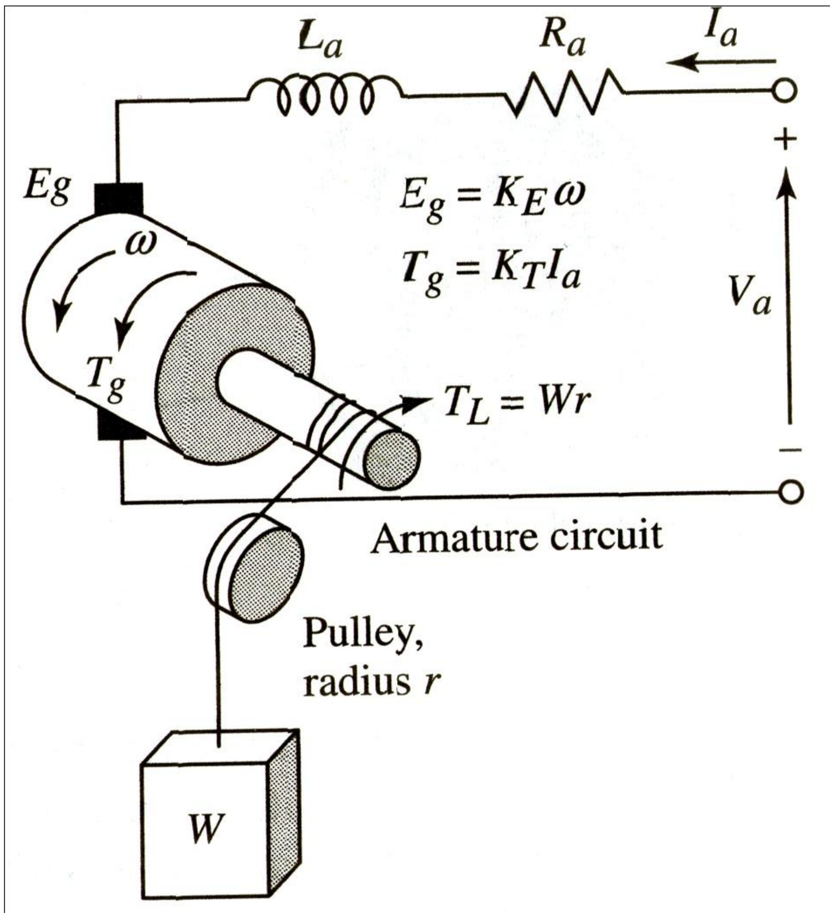
Since, value of La is very small, we can neglect it,

$$Va = Ra\ Ia + Ke\ \omega$$

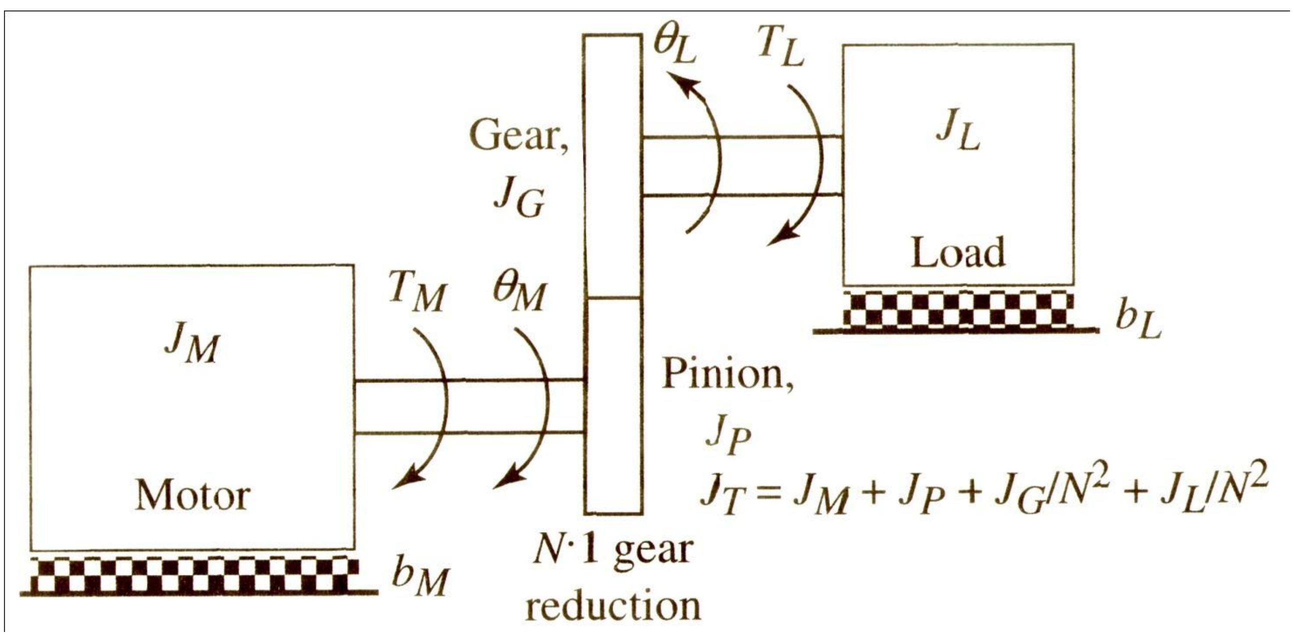
This represents electrical model of a DC motor.

Electromechanical model

The mechanical load on a motor consists of the inertia and the constant torque due to friction or gravity.



K_T = torque constant,
I = current,
J = Polar MI of motor and load,
T_f = Friction torque opposing armature rotation



Consequently, the total torque is given as

$$T = K_{\mathrm{T}} I = J \frac{\mathrm{d}\omega}{\mathrm{d}t} + T_{\mathrm{f}}$$

Taking the Laplace transformation of both sides of the equation

$$K_{\mathrm{T}} I(s) = Js\omega(s) + T_{\mathrm{f}}(s)$$

$$\therefore \; I(s) = \frac{Js\omega(s) + T_{\mathrm{f}}(s)}{K_{\mathrm{T}}}$$

But,

$$V = L\frac{\mathrm{d}I}{\mathrm{d}t} + RI + E$$

Taking the Laplace transformation of both sides of the equation

$$V(s) = LsI(s) + RI(s) + E(s)$$

$$V(s) = Ls\left(\frac{Js\omega(s) + T_{\mathrm{f}}(s)}{K_{\mathrm{T}}}\right) + R\left(\frac{Js\omega(s) + T_{\mathrm{f}}(s)}{K_{\mathrm{T}}}\right) + K_{\mathrm{E}}\omega(s)$$

$$V(s) = \left(\frac{LJs^2}{K_{\mathrm{T}}} + \frac{RJs}{K_{\mathrm{T}}} + K_{\mathrm{E}}\right)\omega(s) + \frac{LT_{\mathrm{f}}(s)s}{K_{\mathrm{T}}} + \frac{RT_{\mathrm{f}}(s)}{K_{\mathrm{T}}}$$

Dividing throughout by K_E

$$\frac{V(s)}{K_{\mathrm{E}}} = \left(\frac{LJs^2}{K_{\mathrm{T}}K_{\mathrm{E}}} + \frac{RJs}{K_{\mathrm{T}}K_{\mathrm{E}}} + 1\right)\omega(s) + \frac{R}{K_{\mathrm{T}}K_{\mathrm{E}}}\left(\frac{Ls}{R} + 1\right)T_{\mathrm{f}}(s)$$

Let us define;

$$K_{\mathrm{m}} = \frac{K_{\mathrm{T}}}{\sqrt{R}}; \quad or\ K_{\mathrm{m}}^{2} = \frac{K_{\mathrm{T}}^{2}}{R} \text{ and let } K_{\mathrm{T}} = K_{\mathrm{E}}$$

$$\frac{V(s)}{K_{\mathrm{E}}} = \left(\frac{LJs^{2}}{RK_{\mathrm{m}}^{2}} + \frac{RJs}{RK_{\mathrm{m}}^{2}} + 1\right)\omega(s) + \frac{R}{K_{\mathrm{T}}^{2}}\left(\frac{Ls}{R} + 1\right)T_{\mathrm{f}}(s)$$

Let,

$$\tau_{\mathrm{e}} = \frac{L}{R}; \quad \tau_{\mathrm{m}} = \frac{J}{K_{\mathrm{m}}^{2}},$$

Where,

τ_{m} and τ_{e} are

Mechanical and electrical time constants of motor , Then the dynamic model can be,

$$\frac{V(s)}{K_E} = \left(\tau_e \tau_m s^2 + \tau_m s + 1\right)\omega(s) + \frac{1}{{K_m}^2}(\tau_e s + 1)T_f(s)$$

For most d.c. motors, $\tau_e = 0$, and hence

$$\frac{V(s)}{K_E} = (\tau_m s + 1)\omega(s) + \frac{1}{{K_m}^2}T_f(s)$$

Rearranging,

$$(\tau_m s + 1)\,\omega(s) = \frac{V(s)}{K_E} - \frac{T_f(s)}{{K_m}^2}$$

By ignoring T_f,

The electromechanical model of a DC motor can be given by the following transfer function,

$$\frac{V(s)}{K_E} = (\tau_m s + 1)\,\omega(s)$$

Derivation for maximum Power output

The torque generated by a DC motor, Tg is directly proportional to the armature current,

$$T_g = K_T \,.\, I_a \qquad \text{--------(1)}$$

where K_T = torque constant

Ia = armature current

using,

$$V_a = R_a I_a + K_E \omega$$

$$= R_a \frac{T_g}{K_T} + K_E \omega$$

$$\frac{T_g}{K_T} R_a = V_a - K_E \omega$$

$$T_g = (V_a - K_E \omega) \frac{K_T}{R_a}$$

$$T_g = V_a \left(\frac{K_T}{R_a}\right) - \left(\frac{K_E K_T}{R_a}\right) \omega \quad \text{———} (2)$$

This is a linear relation,
which is used as a basis
for generating speed-torque
Curves of a DC motor.

By setting $\omega = 0$ in equation(2)
we can obtain the blocked-rotor
torque at the rated Voltage V_a.

This torque is called as stall torque.
& is denoted by T_s.

from (2),

$$T_S = V_a\left(\frac{K_T}{R_a}\right) \quad \text{——— (3)}$$

Theoretical no-load speed is obtained by equating eqn (2) to zero

$$\therefore \frac{V_a K_T}{R_a} = \frac{K_E \omega K_T}{R_a}$$

$$\therefore \omega_0 = \frac{V_a}{K_E} \quad \text{——— (4)}$$

from ① & ③

$$T_g = T_s - \left(\frac{K_e K_T}{R_a}\right) \omega$$

$$\therefore \frac{T_g}{T_s} = 1 - \frac{\left(\frac{K_e K_T}{R_a}\right) \omega}{T_s}$$

$$\therefore \frac{T_g}{T_s} = 1 - \frac{\omega}{\omega_0} \Rightarrow$$

$$\therefore \frac{T_g}{T_s} + \frac{\omega}{\omega_0} = 1 \quad \text{--- (5)}$$

$$\frac{\frac{K_e K_T}{R_a}}{T_s}$$

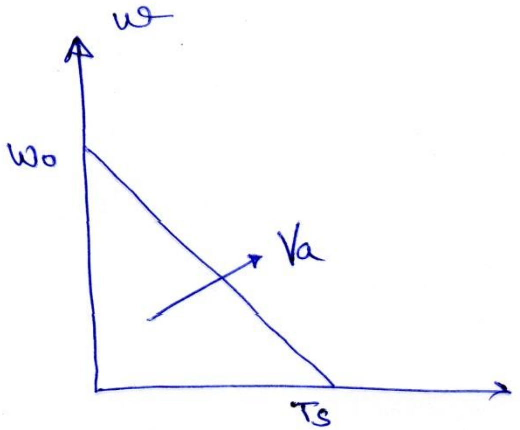
$$= \frac{K_e K_T}{R_a \cdot T_s}$$

$$= \frac{K_e K_T}{R_a \times V_a \frac{K_T}{R_a}}$$

$$= \frac{K_e}{V_a}$$

$$= \frac{1}{\omega_o}$$

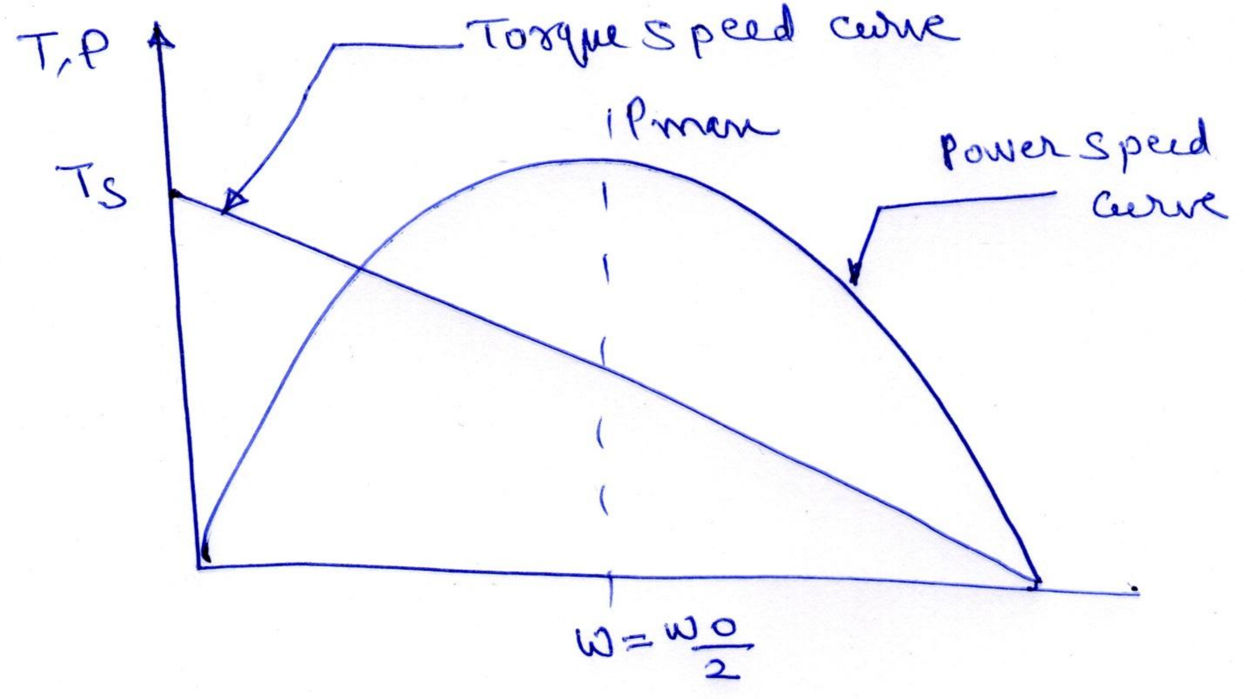
A curve plot from the equation (5)



$\rightarrow T_g$

Variation of generated torque with ang. vel.

If the motor rotates at constant speed, it produces a parabolic mechanical power curve,



Power variation in DC motor

Power delivered by motor,

$$P = T_g \omega$$

Now, $$\frac{T_g}{T_s} = \left(1 - \frac{\omega}{\omega_0}\right)$$

$$T_g = T_s\left(1 - \frac{\omega}{\omega_0}\right)$$

$$\therefore \; P = T_s \omega \left(1 - \frac{\omega}{\omega_0}\right)$$

for maximum power,

$$\frac{dP}{d\omega} = 0$$

$$\therefore T_S - \frac{2\omega}{\omega_0} T_S = 0$$

$$\therefore 1 = \frac{2\omega}{\omega_0}$$

$$\therefore \omega_0 = 2\omega$$

$$\& \ \omega = \frac{\omega_0}{2}$$

HEAT DISSIPATION IN DC MOTOR

- The main factor that limits the performance of a DC motor is
  - Heat dissipation in the armature

Let ,

Wc = energy dissipated in a motor

tc = time for which the motor rotates (seconds)

Now,

$$P = \frac{dw}{dt}$$

- P = time rate of work done

$$Wc = \int_{0}^{t_c} P(t)\,dt \qquad \text{----------------(1)}$$

HEAT DISSIPATION IN DC MOTOR

- Also , $P = I^2 R$

$$W_c = R \int_0^{t_c} I^2 (t)\, dt$$

$$W_c = R\, I^2 \int_0 (t)\, dt$$

$$W_c = R I^2 t_c \qquad \text{---------------(2)}$$

- Also , T α I
- $T = K_T I$

$$I^2 = \frac{T_L^2}{K_T^2} \qquad \text{---------------(3)}$$

$$W_{C(T_L)} = I_a^2 R t_C = \frac{R}{K_T^2} T_L^2 t_C$$

HEAT DISSIPATION IN DC MOTOR

Factors that influence heat dissipation:

- Two factors are:
  - Velocity profile
  - Coupling ratio

- **Velocity profile:**
  - It refers to the way in which the angular velocity of the motor is varied with time as the load moves.

- **Coupling ratio:**
  - It refers to the ratio of angular velocity of the motor inertia J_M to the velocity of the load inertia J_L

- The optimum velocity profile w(t) and Coupling ratio Go may be determined to minimise Wc.

VELOCITY PROFILE OPTIMIZATION

Three possible velocity profile for DC motor:

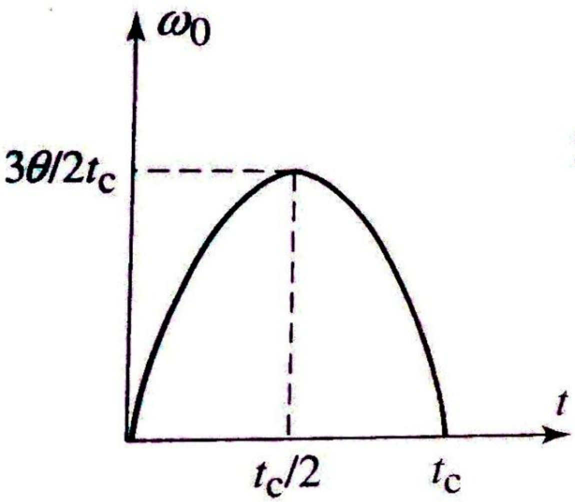
- Parabolic
- Triangular
- Trapezoidal

- The effect of friction is neglected while finding optimum VP.

- It is assumed that friction is independent of velocity.

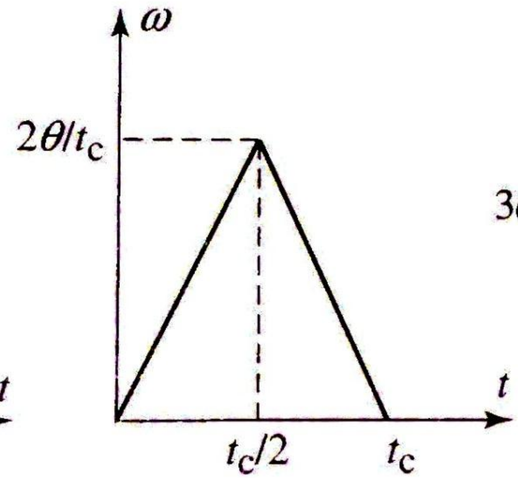
VELOCITY PROFILE OPTIMIZATION

Three possible velocity profile for DC motor:



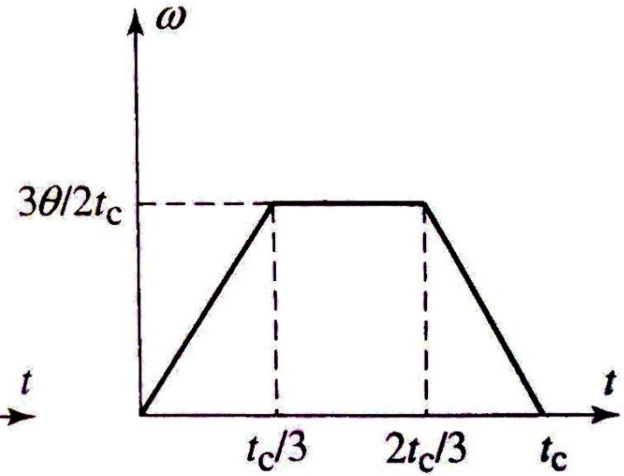
(a)

Parabolic



(b)

Triangular



(c)

Trapezoidal

- Load accelerated smoothly
- Load decelerated smoothly

- Load accelerated at fixed rate
- Load decelerated at same rate

- 3 equal time zones
  - Acceleration zone
  - Run time
  - Deceleration zone

VELOCITY PROFILE OPTIMIZATION

- The parabolic velocity profile is given by

$$\omega(t) = 6\theta \frac{t_C - t}{t_C^3} t$$

If we substitute the three velocity profile in the equation

$$Wc = \int_0^{t_c} P(t)\,dt \qquad \text{---------------(1)}$$

Then ,

Energy dissipation during tc can be expressed as,

$$W_C = \frac{R}{K_T^2}\left[\lambda \frac{J_T^2 \theta^2}{t_C^3} + T_L^2 t_C\right] \qquad \text{where,} \qquad \boldsymbol{\lambda = 12/\eta}$$

η is the velocity profile efficiency, $\qquad \eta = \dfrac{W_{C0}}{W_C}$

INERTIA MATCHING

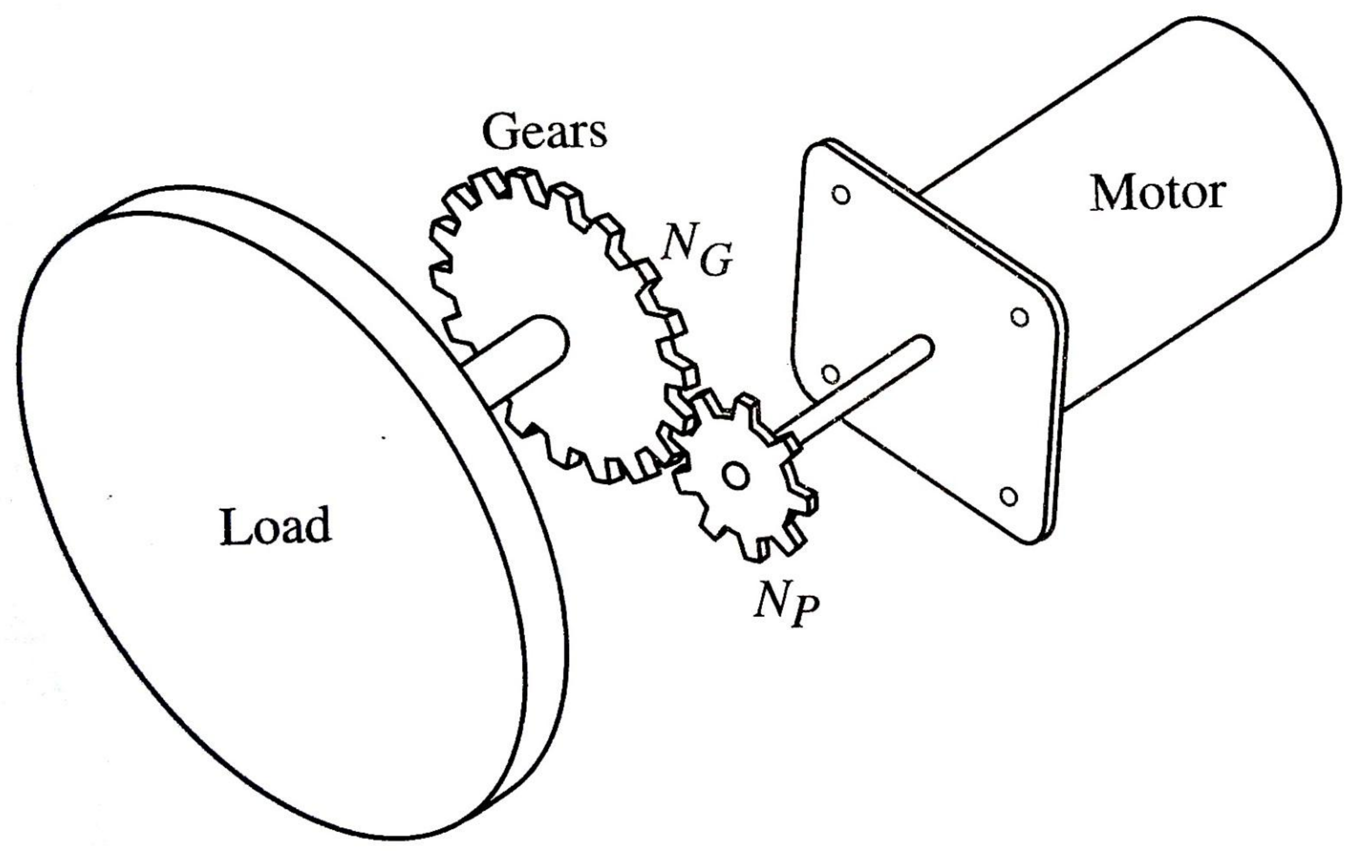
Three possible ways to couple motor to a load are:

- Gear Transmission
- Belt pulley drive
- Lead screw drive

INERTIA MATCHING

Gear Transmission

- A DC motor driving a load via a gear reducer unit is shown.



INERTIA MATCHING

Gear Transmission

Let,

$\theta_L' =$ Angular rotation,

$J_L =$ Load inertia,

$T_L =$ Load torque

- These load parameters are related to the motor shaft by the following relations:

$$\theta_L' = N\theta_L, \qquad J_L' = J_L/N^2, \qquad \text{and} \qquad T_L' = T_L/N$$

where, $N = N_G/N_P$

is the ratio of number of teeth to the gear to the number of teeth to the pinion.

INERTIA MATCHING

Gear Transmission

- The energy dissipation equation

$$W_C = \frac{R}{K_T^2}\left[\lambda\frac{J_T^2\theta^2}{t_C^3} + T_L^2 t_C\right]$$

becomes

$$W_C = \frac{R}{K_T^2}\frac{12}{\eta}\frac{J_L^2\theta_L^2}{t_C^3}\left[N^2\left(\frac{J_M}{J_L} + \frac{1}{N^2}\right)^2 + \frac{\gamma}{N^2}\right] \text{ ------------(A)}$$

since $J_M = J_L / N^2$

INERTIA MATCHING

Gear Transmission

$$\gamma = \frac{\eta}{12}\left[\frac{T_L t_C^2}{\theta_L J_L}\right]^2$$

- In order to minimize the heat dissipation in the motor for a given load,
- Differentiating equation (A) w.r.t. N^2 and equating to zero

$$N_0^2 = \frac{J_L}{J_M}\sqrt{1+\gamma} \quad \text{------------(B)}$$

INERTIA MATCHING

Gear Transmission

- In the absence of load torque,

$$T_L = 0, \text{ so } \gamma = 0,$$

equation (B) becomes

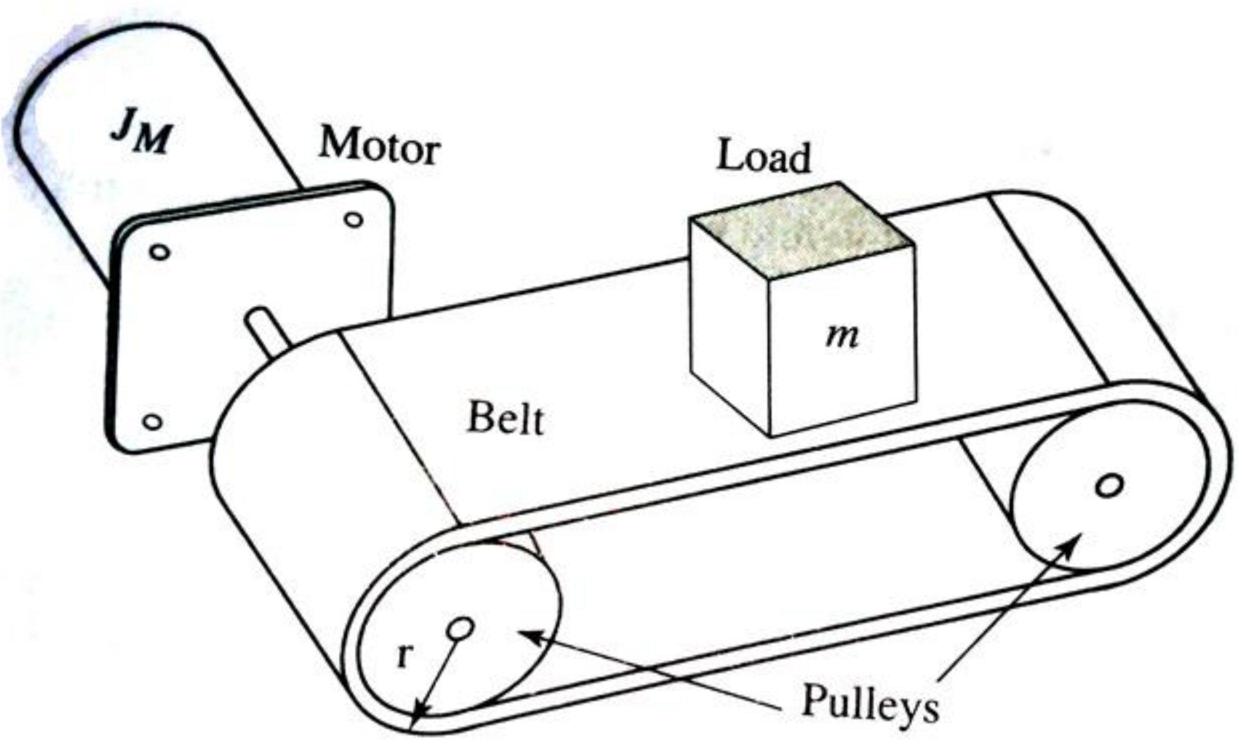
$$N_0 = \sqrt{J_L / J_M}$$

This is known as inertia matching.

INERTIA MATCHING

Belt pulley drive

- It consists of a motor turning a pulley.
- The pulley pulls a belt to which a load is attached.
- Thus a rotary motion is converted to a translatory motion.



INERTIA MATCHING

Belt pulley drive

- The coupling ratio G is the reciprocal of the radius r of the pulley connected to the motor.

Let,

- m = load to be moved at a distance x, in time tc.
- F= opposing force

INERTIA MATCHING

Belt pulley drive

- The energy dissipation in the motor armature during time tc is

$$W_C = \frac{R}{K_T^2}\frac{12}{\eta}\frac{m^2x^2}{t_C^3}\left[G^2\left(\frac{J_M}{m} + \frac{1}{G^2}\right)^2 + \frac{\beta}{G^2}\right]$$

- where

$$\beta = \frac{\eta}{12}\left[\frac{Ft_C^2}{mx}\right]^2$$

INERTIA MATCHING

Belt pulley drive

- The optimum coupling is easily determined by

$$G_O^2 = \frac{m}{J_M}\sqrt{1 + \beta}$$

- Thus the optimum pulley radius is

$$r_O = \sqrt{J_M/m\sqrt{1 + \beta}}$$

- If the load F = 0, then $\beta = 0$

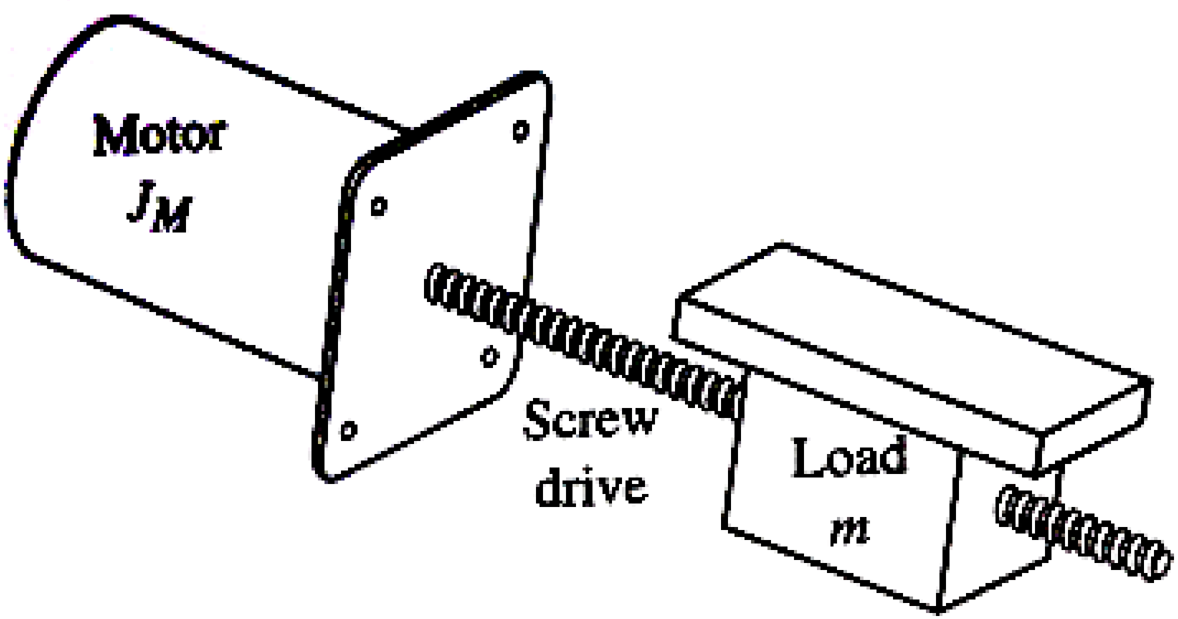
- Therefore an exact inertia match is

$$r_O = \sqrt{J_M/m}$$

INERTIA MATCHING

Lead screw drive

- A typical lead screw drive is shown in figure.



INERTIA MATCHING

Lead screw drive

- The pitch is taken as the number of revolutions per unit length.

$$P = G/2\pi$$

- As, G = motor rotation in radians per unit length,

- The optimum pitch for inertia matching is

$$P_{opt} = (1/2\pi)\sqrt{m/J_M}$$