Modeling of DC motor

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The impedance of a DC motor’s armature can be modelled as a resistance $R$ in series with the parallel combination of an inductance $L$ and a second resistance $RL$. 
• However, as the conducting armature begins to rotate in the magnetic field produced by the stator, a voltage called the back Emf $E_g$ is induced in the armature windings opposing the applied voltage.

• The back Emf is proportional to the angular velocity of the rotor $\omega$ in rad/sec:

$$E_g = K_e \omega$$  \hspace{1cm} (1)

• where, the proportionality constant $K_e$ is called the voltage constant of the motor.
Applying Kirchoff’s voltage law,

\[
Va = \frac{La}{dt} \frac{dI_a}{dt} + Ra I_a + Eg + Ke \omega
\]

where,
La = inductance of armature,
Ia = armature current,
Va = armature voltage
Ra = armature resistance
Since, value of $L_a$ is very small, we can neglect it,

$$V_a = R_a I_a + K_e \omega$$

This represents electrical model of a DC motor.
Electromechanical model
The mechanical load on a motor consists of the inertia and the constant torque due to friction or gravity.
\( K_T = \) torque constant,
\( I = \) current,
\( J = \) Polar MI of motor and load,
\( T_f = \) Friction torque opposing armature rotation

\[
J_T = J_M + J_P + \frac{J_G}{N^2} + \frac{J_L}{N^2}
\]
Consequently, the total torque is given as

\[ T = K_T I = J \frac{d\omega}{dt} + T_f \]

Taking the Laplace transformation of both sides of the equation

\[ K_T I(s) = J s \omega(s) + T_f(s) \]

\[ \therefore I(s) = \frac{J s \omega(s) + T_f(s)}{K_T} \]
But,

\[ V = L \frac{dl}{dt} + RI + E \]

Taking the Laplace transformation of both sides of the equation

\[ V(s) = LsI(s) + RI(s) + E(s) \]

\[ V(s) = Ls \left( \frac{Js\omega(s) + T_f(s)}{K_T} \right) + R \left( \frac{Js\omega(s) + T_f(s)}{K_T} \right) + K_E\omega(s) \]

\[ V(s) = \left( \frac{LJ^2}{K_T} + \frac{RJS}{K_T} + K_E \right)\omega(s) + \frac{LTF(s)s}{K_T} + \frac{RT_f(s)}{K_T} \]
Dividing throughout by $K_E$

$$\frac{V(s)}{K_E} = \left( \frac{LJ_s^2}{K_T K_E} + \frac{RJ_s}{K_T K_E} + 1 \right) \omega(s) + \frac{R}{K_T K_E} \left( \frac{L_s}{R} + 1 \right) T_f(s)$$
Let us define;

\[ K_m = \frac{K_T}{\sqrt{R}}; \quad \text{or} \quad K_m^2 = \frac{K_T^2}{R} \quad \text{and let} \quad K_T = K_E \]

\[
\frac{V(s)}{K_E} = \left( \frac{LJ_s^2}{RK_m^2} + \frac{RJs}{RK_m^2} + 1 \right)\omega(s) + \frac{R}{K_T^2} \left( \frac{Ls}{R} + 1 \right)T_f(s)
\]

Let,

\[
\tau_e = \frac{L}{R}; \quad \tau_m = \frac{J}{K_m^2},
\]

Where,

\( \tau_m \) and \( \tau_e \) are

Mechanical and electrical time constants of motor.
Then the dynamic model can be,

\[ \frac{V(s)}{K_E} = (\tau_e \tau_m s^2 + \tau_m s + 1) \omega(s) + \frac{1}{K_m^2} (\tau_e s + 1) T_f(s) \]

For most d.c. motors, \( \tau_e = 0 \), and hence

\[ \frac{V(s)}{K_E} = (\tau_m s + 1) \omega(s) + \frac{1}{K_m^2} T_f(s). \]
Rearranging,

\[(\tau_m s + 1) \omega(s) = \frac{V(s)}{K_E} - \frac{T_f(s)}{K_m^2}\]

By ignoring \(T_f\),

The electromechanical model of a DC motor can be given by the following transfer function,

\[\frac{V(s)}{K_E} = (\tau_m s + 1) \omega(s)\]
Derivation for maximum Power output

The torque generated by a DC motor, $T_g$ is directly proportional to the armature current,

$$T_g = K_T \cdot I_a$$  \hspace{1cm} (1)

where $K_T =$ torque constant
$I_a =$ armature current
Using,

\[ V_a = R_a I_a + K E \omega \]

\[ = R_a \frac{T_g}{K_t} + K E \omega \]

\[ T_g \frac{R_a}{K_t} = V_a - K E \omega \]

\[ T_g = (V_a - K E \omega) \frac{K_t}{R_a} \]

\[ T_g = V_a \left( \frac{K_t}{R_a} \right) - \left( \frac{K E}{R_a} \right) \omega \quad (2) \]
This is a linear relation, which is used as a basis for generating speed-torque curves of a DC motor.

By setting $\omega = 0$ in equation (2), we can obtain the blocked-rotor torque at the rated voltage $V_a$.

This torque is called as stall torque. It is denoted by $T_s$. 
Theoretical no-load speed is obtained by equating Eqn (2) to zero.

\[
\omega_0 = \frac{V_a}{\frac{K E}{K R}} \quad (4)
\]
\[ T_g = T_s - \left( \frac{K e K T}{R a} \right) \omega \]

\[ \frac{T_g}{T_s} = 1 - \left( \frac{K e K T}{R a} \right) \frac{\omega}{\omega_0} \]

\[ \frac{T_g}{T_s} + \frac{\omega}{\omega_0} = 1 \quad \text{(5)} \]
\[ \frac{k e K T}{R_a} \]
\[ \frac{1}{T_s} \]
\[ = \frac{k e K T}{R_a \cdot T_s} \]
\[ = \frac{k e K T}{R_a \cdot V_a \cdot K T \cdot R_a} \]
\[ = \frac{k e}{V_a} \]
\[ = \frac{1}{\omega_0} \]
A curve plot from the equation (5).

Variation of generated torque with angular vel.
If the motor rotates at constant speed, it produces a parabolic mechanical power curve.

Torque Speed curve

Power Speed curve

\[ w = \frac{w_0}{2} \]

Power variation in DC motor
Power delivered by motor,

\[ P = T_g \omega \]

Now,

\[ \frac{T_g}{T_s} = \left( 1 - \frac{\omega}{\omega_0} \right) \]

\[ T_g = T_s \left( 1 - \frac{\omega}{\omega_0} \right) \]

\[ P = T_s \omega \left( 1 - \frac{\omega}{\omega_0} \right) \]
For maximum power,

\[ \frac{dP}{d\omega} = 0 \]

\[ T_s - \frac{2\omega}{\omega_0} T_s = 0 \]

\[ \therefore 1 = \frac{2\omega}{\omega_0} \]

\[ \therefore \omega_0 = \frac{2\omega}{3} \]

\[ \& \, \omega = \frac{\omega_0}{2} \]
HEAT DISSIPATION IN DC MOTOR

• The main factor that limits the performance of a DC motor is
  • Heat dissipation in the armature

Let,
  \( W_c = \) energy dissipated in a motor
  \( t_c = \) time for which the motor rotates (seconds)

Now,

\[
P = \frac{dw}{dt}
\]

• \( P = \) time rate of work done

\[
W_c = \int_{0}^{t_c} P(t) \, dt \quad \text{-------------(1)}
\]
HEAT DISSIPATION IN DC MOTOR

- Also, \( P = I^2 R \)

\[
W_c = R \int_0^{t_c} I^2(t) \, dt
\]

\[
W_c = R I^2 t_c
\]  \hspace{1cm} \text{-------------------}(2)

- Also, \( T \propto I \)
- \( T = K_T I \)

\[
I^2 = \frac{T_L^2}{K_T^2}
\]  \hspace{1cm} \text{-------------------}(3)

\[
W_{C(T_L)} = I_a^2 R t_C = \frac{R}{K_T^2} T_L^2 t_C
\]
HEAT DISSIPATION IN DC MOTOR

Factors that influence heat dissipation:
• Two factors are:
  • Velocity profile
  • Coupling ratio

• Velocity profile:
  • It refers to the way in which the angular velocity of the motor is varied with time as the load moves.

• Coupling ratio:
  • It refers to the ratio of angular velocity of the motor inertia $J_M$ to the velocity of the load inertia $J_L$

• The optimum velocity profile $w(t)$ and Coupling ratio $G_o$ may be determined to minimise $W_c$. 
VELOCITY PROFILE OPTIMIZATION

Three possible velocity profile for DC motor:

• Parabolic
• Triangular
• Trapezoidal

• The effect of friction is neglected while finding optimum VP.

• It is assumed that friction is independent of velocity.
VELOCITY PROFILE OPTIMIZATION

Three possible velocity profile for DC motor:

- **Parabolic**
  - Load accelerated smoothly
  - Load decelerated smoothly

- **Triangular**
  - Load accelerated at fixed rate
  - Load decelerated at same rate

- **Trapezoidal**
  - 3 equal time zones
  - Acceleration zone
  - Run time
  - Deceleration zone
VELOCITY PROFILE OPTIMIZATION

• The parabolic velocity profile is given by

\[ \omega(t) = 6\theta \frac{t_C - t}{t_C^3} \]

If we substitute the three velocity profile in the equation

\[ W_C = \int_0^{t_c} P(t) \, dt \quad \text{----------------(1)} \]

Then,

Energy dissipation during tc can be expressed as,

\[ W_C = \frac{R}{K_T^2} \left[ \lambda \frac{J_T^2 \theta^2}{t_C^3} + T_L^2 t_C \right] \quad \text{where}, \quad \lambda = \frac{12}{\eta} \]

\[ \eta \text{ is the velocity profile efficiency,} \quad \eta = \frac{W_{C0}}{W_C} \]
INERTIA MATCHING

Three possible ways to couple motor to a load are:

• Gear Transmission

• Belt pulley drive

• Lead screw drive
INERTIA MATCHING

Gear Transmission

- A DC motor driving a load via a gear reducer unit is shown.
INERTIA MATCHING

Gear Transmission

Let,
\[
\theta'_L = \text{Angular rotation}, \\
J'_L = \text{Load inertia}, \\
T'_L = \text{Load torque}
\]

• These load parameters are related to the motor shaft by the following relations:
\[
\theta'_L = N\theta_L, \quad J'_L = J_L/N^2, \quad \text{and} \quad T'_L = T_L/N
\]
where,
\[
N = N_G/N_P
\]
is the ratio of number of teeth to the gear to the number of teeth to the pinion.
INERTIA MATCHING

Gear Transmission

• The energy dissipation equation

\[ W_C = \frac{R}{K_T^2} \left[ \frac{J_T^2 \theta^2}{t_C^3} + T_L^2 t_C \right] \]

becomes

\[ W_C = \frac{R}{K_T^2} \frac{12 J_L^2 \theta_L^2}{\eta t_C^3} \left[ N^2 \left( \frac{J_M}{J_L} + \frac{1}{N^2} \right)^2 + \frac{\gamma}{N^2} \right] \]  

\[ \text{since } J_M = J_L/N^2 \]
INERTIA MATCHING

Gear Transmission

\[ \gamma = \frac{\eta}{12} \left[ \frac{T_L t^2_C}{\theta_L J_L} \right]^2 \]

- In order to minimize the heat dissipation in the motor for a given load,
- Differentiating equation (A) w.r.t. \( N^2 \) and equating to zero

\[ N_0^2 = \frac{J_L}{J_M} \sqrt{1 + \gamma} \quad \text{---------(B)} \]
INERTIA MATCHING

Gear Transmission

• In the absence of load torque,

\[ T_L = 0, \text{ so } \gamma = 0, \]

equation (B) becomes

\[ N_0 = \sqrt{J_L/J_M} \]

This is known as inertia matching.
INERTIA MATCHING

Belt pulley drive

- It consists of a motor turning a pulley.
- The pulley pulls a belt to which a load is attached.
- Thus a rotary motion is converted to a translatory motion.
INERTIA MATCHING

Belt pulley drive

• The coupling ratio $G$ is the reciprocal of the radius $r$ of the pulley connected to the motor.

Let,

• $m =$ load to be moved at a distance $x$, in time $t_c$.
• $F =$ opposing force
INERTIA MATCHING

Belt pulley drive

• The energy dissipation in the motor armature during time \( t_c \) is

\[
W_c = \frac{R}{K_T^2} \frac{12}{\eta} \frac{m^2 x^2}{t_c^3} \left[ G^2 \left( \frac{J_M}{m} + \frac{1}{G^2} \right)^2 + \frac{\beta}{G^2} \right]
\]

• where

\[
\beta = \frac{\eta}{12} \left[ \frac{F t_c^2}{m x} \right]^2
\]
INERTIA MATCHING

Belt pulley drive

• The optimum coupling is easily determined by

\[ G_o^2 = \frac{m}{J_M} \sqrt{1 + \beta} \]

• Thus the optimum pulley radius is

\[ r_o = \sqrt{\frac{J_M}{m} \sqrt{1 + \beta}} \]

• If the load \( F = 0 \), then \( \beta = 0 \)

• Therefore an exact inertia match is

\[ r_o = \sqrt{\frac{J_M}{m}} \]
INERTIA MATCHING

Lead screw drive

- A typical lead screw drive is shown in figure.
INERTIA MATCHING

Lead screw drive

- The pitch is taken as the number of revolutions per unit length.

\[ P = \frac{G}{2\pi} \]

- As, \( G = \) motor rotation in radians per unit length,

- The optimum pitch for inertia matching is

\[ P_{\text{opt}} = \frac{1}{2\pi} \sqrt{\frac{m}{J_M}} \]