Engineering Mechanics- Distributed Forces

Mass Moment of Inertia

- Encounter in the engineering problems involving rotational motion of rigid bodies in dynamics

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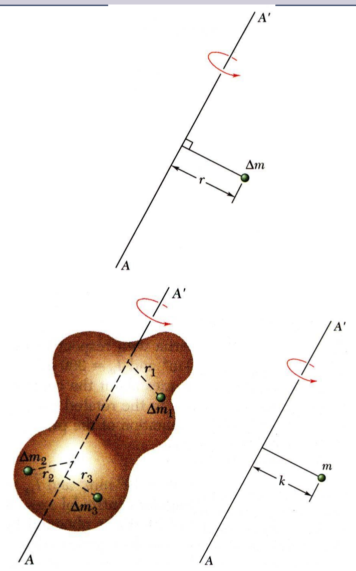
Moment of Inertia of a Mass



- Angular acceleration about the axis *AA'* of the small mass Δm due to the application of a couple is related to $r^2\Delta m$.

  $r^2\Delta m$ = *moment of inertia* of the mass Δm with respect to the axis *AA'*

- For a body of mass *m* the resistance to rotation about the axis *AA' is*

$$I = r_1^2\Delta m + r_2^2\Delta m + r_3^2\Delta m + \cdots$$

$$= \int r^2 dm = \textit{mass moment of inertia}$$

- The radius of gyration for a concentrated mass with equivalent mass moment of inertia is

$$I = k^2 m \qquad k = \sqrt{\frac{I}{m}}$$

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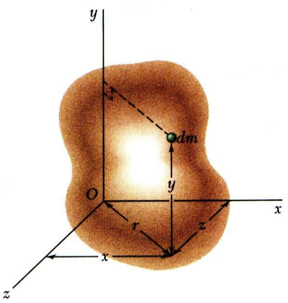
Moment of Inertia of a Mass



- Moment of inertia with respect to the *y* coordinate axis is

$$I_y = \int r^2 dm = \int \left(z^2 + x^2\right) dm$$

- Similarly, for the moment of inertia with respect to the *x* and *z* axes,

$$I_x = \int \left(y^2 + z^2\right) dm$$
$$I_z = \int \left(x^2 + y^2\right) dm$$

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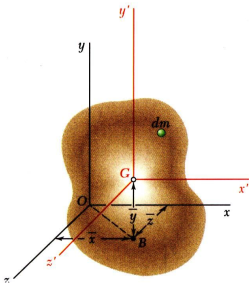
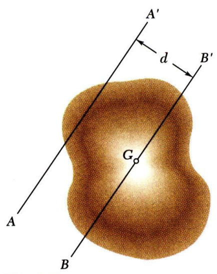
Parallel Axis Theorem





- For the rectangular axes with origin at *O* and parallel centroidal axes,

$$I_x = \int \left(y^2 + z^2\right) dm = \int \left[\left(y' + \bar{y}\right)^2 + \left(z' + \bar{z}\right)^2\right] dm$$
$$= \int \left(y'^2 + z'^2\right) dm + 2\bar{y} \int y' dm + 2\bar{z} \int z' dm + \left(\bar{y}^2 + \bar{z}^2\right) \int dm$$

$$I_x = \bar{I}_{x'} + m\left(\bar{y}^2 + \bar{z}^2\right)$$
$$I_y = \bar{I}_{y'} + m\left(\bar{z}^2 + \bar{x}^2\right)$$
$$I_z = \bar{I}_{z'} + m\left(\bar{x}^2 + \bar{y}^2\right)$$

- Generalizing for any axis *AA'* and a parallel centroidal axis,

$$I = \bar{I} + md^2$$

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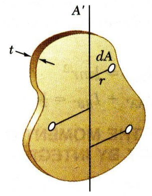
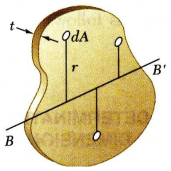
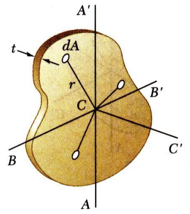
Moments of Inertia of Thin Plates







- For a thin plate of uniform thickness *t* and homogeneous material of density *r*, the mass moment of inertia with respect to axis *AA'* contained in the plate is

$$I_{AA'} = \int r^2 dm = \rho t \int r^2 dA$$
$$= \rho t\, I_{AA',area}$$

- Similarly, for perpendicular axis *BB'* which is also contained in the plate,

$$I_{BB'} = \rho t\, I_{BB',area}$$

- For the axis *CC'* which is perpendicular to the plate,

$$I_{CC'} = \rho t\, J_{C,area} = \rho t \left(I_{AA',area} + I_{BB',area}\right)$$
$$= I_{AA'} + I_{BB'}$$

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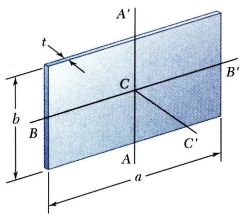
Moments of Inertia of Thin Plates

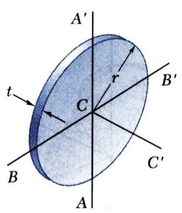


- For the principal centroidal axes on a rectangular plate,

$$I_{AA'} = \rho t I_{AA',area} = \rho t\left(\tfrac{1}{12} a^3 b\right) = \tfrac{1}{12} m a^2$$

$$I_{BB'} = \rho t I_{BB',area} = \rho t\left(\tfrac{1}{12} a b^3\right) = \tfrac{1}{12} m b^2$$

$$I_{CC'} = I_{AA',mass} + I_{BB',mass} = \tfrac{1}{12} m\left(a^2 + b^2\right)$$



- For centroidal axes on a circular plate,

$$I_{AA'} = I_{BB'} = \rho t I_{AA',area} = \rho t\left(\tfrac{1}{4} \pi r^4\right) = \tfrac{1}{4} m r^2$$

$$I_{CC'} = I_{AA'} + I_{BB'} = \tfrac{1}{2} m r^2$$

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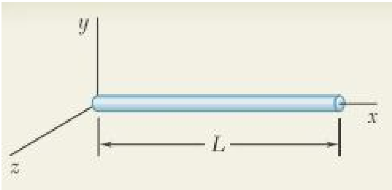
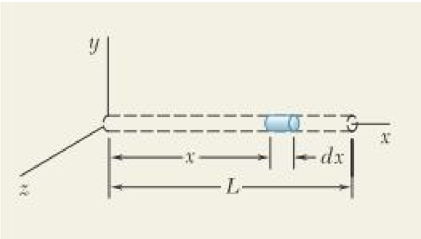
Mass Moment of Inertia of a Slendor Rod

Determine the moment of inertia of a slender rod of length L and mass m with respect to an axis which is perpendicular to the rod and passes through one end of the rod.





$$I_z = \int r^2 dm$$

$$I_z = \int_0^L x^2 (\bar{m} dx)$$

$$\bar{m} = mass / unit\ length = \frac{m}{L}$$

$$= \frac{\bar{m}L^3}{3} = \frac{(\bar{m}L)L^2}{3} = \frac{mL^2}{3}$$

$$(I_z)_{cg} = I_z - m\left(\frac{L}{2}\right)^2 = \frac{mL^2}{12}$$

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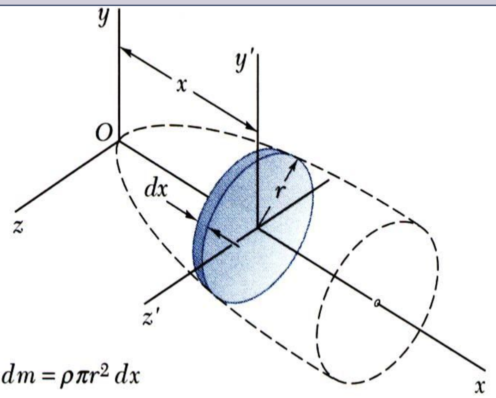
Moments of Inertia of a 3D Body by Integration



$$dm = \rho \pi r^2\, dx$$

$$dI_x = \tfrac{1}{2} r^2\, dm$$

$$dI_y = dI_{y'} + x^2\, dm = \left(\tfrac{1}{4} r^2 + x^2\right) dm$$

$$dI_z = dI_{z'} + x^2\, dm = \left(\tfrac{1}{4} r^2 + x^2\right) dm$$

- Moment of inertia of a homogeneous body is obtained from double or triple integrations of the form

$$I = \rho \int r^2 dV$$

- For bodies with two planes of symmetry, the moment of inertia may be obtained from a single integration by choosing thin slabs perpendicular to the planes of symmetry for *dm*.

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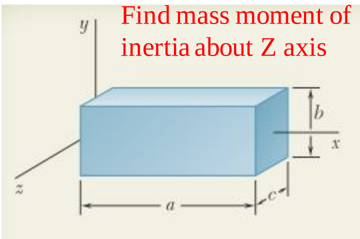
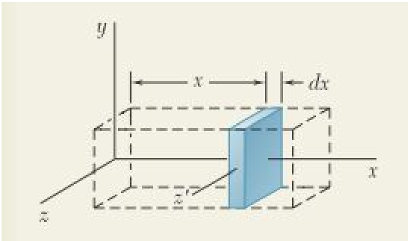
Mass Moment of Inertia of a Rectangular Prism

Find mass moment of inertia about Z axis





$$I_z = \tfrac{1}{12}m(4a^2 + b^2)$$

$$dm = \rho bc\, dx$$

$$dI_{z'} = \tfrac{1}{12}b^2\, dm$$

$$dI_z = dI_{z'} + x^2\, dm$$

$$= \tfrac{1}{12}b^2\, dm + x^2\, dm$$

$$= (\tfrac{1}{12}b^2 + x^2)\rho bc\, dx$$

$$I_z = \int dI_z = \int_0^a (\tfrac{1}{12}b^2 + x^2)\rho bc\, dx$$

$$= \rho abc(\tfrac{1}{12}b^2 + \tfrac{1}{3}a^2)$$

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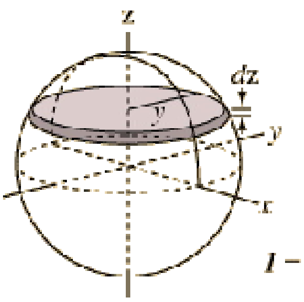
Mass Moment of Inertia of a Sphere



$$dI = \frac{1}{2}y^2 dm = \frac{1}{2}y^2 \rho dV = \frac{1}{2}y^2 \rho\pi y^2 dz$$

and the integral becomes

$$I = \frac{1}{2}\rho\pi \int_{-R}^{R} y^4 dz = \frac{1}{2}\rho\pi \int_{-R}^{R} (R^2 - z^2)^2 dz = \frac{8}{15}\rho\pi R^5$$

Radius = R
Mass = M
Density = $\rho = \dfrac{M}{V} = \dfrac{M}{\frac{4}{3}\pi R^3}$

Substituting the density expression gives

$$I = \frac{8}{15}\left[\frac{M}{\frac{4}{3}\pi R^3}\right]\pi R^5 = \frac{2}{5}MR^2$$

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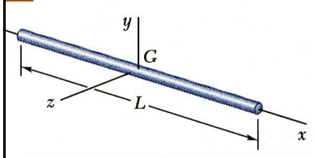
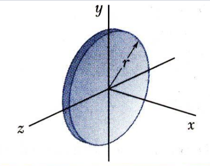
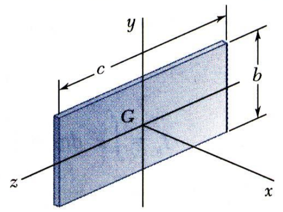
Moments of Inertia of Common Geometric Shapes



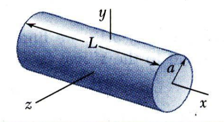
$$I_y = I_z = \tfrac{1}{12} mL^2$$



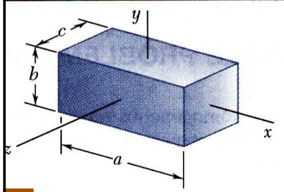
$$I_x = \tfrac{1}{2} mr^2$$
$$I_y = I_z = \tfrac{1}{4} mr^2$$



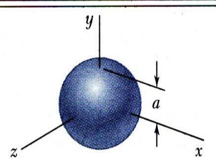
$$I_x = \tfrac{1}{12} m(b^2 + c^2)$$
$$I_y = \tfrac{1}{12} mc^2$$
$$I_z = \tfrac{1}{12} mb^2$$



$$I_x = \tfrac{1}{2} ma^2$$
$$I_y = I_z = \tfrac{1}{12} m(3a^2 + L^2)$$



$$I_x = \tfrac{1}{12} m(b^2 + c^2)$$
$$I_y = \tfrac{1}{12} m(c^2 + a^2)$$
$$I_z = \tfrac{1}{12} m(a^2 + b^2)$$



$$I_x = I_y = I_z = \tfrac{2}{5} ma^2$$

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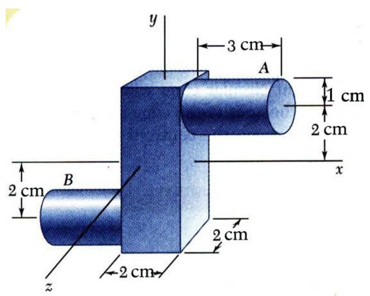
Sample Problem: Composite Section



Determine the moments of inertia of the steel forging with respect to the *xyz* coordinate axes, knowing that the specific weight of steel is 7896 kg/m^3.

SOLUTION:

- With the forging divided into a prism and two cylinders, compute the mass and moments of inertia of each component with respect to the *xyz* axes using the parallel axis theorem.
- Add the moments of inertia from the components to determine the total moments of inertia for the forging.

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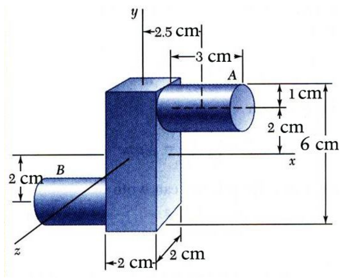
Sample Problem: Composite Section

SOLUTION:

- Compute the moments of inertia of each component with respect to the *xyz* axes.



each cylinder :

$$m = \gamma V = \left(7896 \text{ kg/m}^3\right)\left(\pi \times 1^2 \times 3\right) \times 10^6 \ m^3$$

$$m = 0.0744 \text{ kg}$$

cylinders $(a = 1\text{cm}, L = 3\text{cm}, \bar{x} = 2.5\text{cm}, \bar{y} = 2\text{cm})$:

$$I_x = \tfrac{1}{2} m a^2 + m\bar{y}^2$$
$$= \tfrac{1}{2}(0.0744)(1)^2 + (0.0744)(2)^2$$
$$= 0.5394 \text{ kgm}^2$$

$$I_y = \tfrac{1}{12} m\left[3a^2 + L^2\right] + m\bar{x}^2$$
$$= \tfrac{1}{12}(0.0744)\left[3(1)^2 + (3)^2\right] + (0.0744)(2.5)^2$$
$$= 0.5394 \text{ kgm}^2$$

$$\mathbf{I_z} = \tfrac{1}{12}\mathbf{m}\left[3\mathbf{a}^2 + \mathbf{L}^2\right] + \mathbf{m}\left[\bar{\mathbf{x}}^2 + \bar{\mathbf{y}}^2\right]$$
$$= \tfrac{1}{12}(0.0744)\left[3(1)^2 + (3)^2\right] + (0.0744)\left[(2.5)^2 + (2)^2\right]$$
$$= 0.837 \ kgm^2$$

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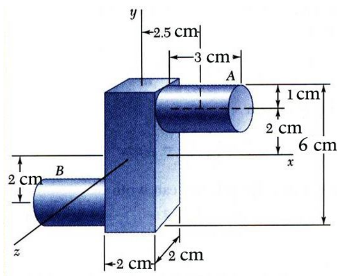
Sample Problem: Composite Section



prism $(a = 2 \text{ cm}, b = 6 \text{ cm}, c = 2 \text{ cm})$:

$$I_x = I_z = \tfrac{1}{12} m\left[b^2 + c^2\right] = \tfrac{1}{12}(0.1895)\left[(6)^2 + (2)^2\right]$$
$$= 0.632 \text{ kgm}^2$$

$$I_y = \tfrac{1}{12} m\left[c^2 + a^2\right] = \tfrac{1}{12}(0.1895)\left[(2)^2 + (2)^2\right]$$
$$= 0.126 \text{ kgm}^2$$

prism :

$$m = \gamma V = \left(7896 \text{ kg/m}^3\right)\left(2 \times 2 \times 6\right) \times 10^{-6} \text{m}^3$$

$$m = 0.1895 \text{ kg}$$

- Add the moments of inertia from the components to determine the total moments of inertia.

$$I_x = 0.632 + 2(0.372)$$
$$\boxed{I_x = 1.376 \text{ kgm}^2}$$

$$I_y = 0.126 + 2(0.5394)$$
$$\boxed{I_y = 1.2048 \text{ kgm}^2}$$

$$I_z = 0.632 + 2(0.837)$$
$$\boxed{I_z = 2.306 \text{ kgm}^2}$$