Fundamentals of Vibration

MECH375G

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Outline

- Why vibration is important?
- Definition; mass, spring (or stiffness) dashpot
- Newton's laws of motion, 2nd order ODE
- Three types of vibration for SDOF sys.
- Alternative way to find eqn of motion: energy methods
- Examples

Why to study vibration

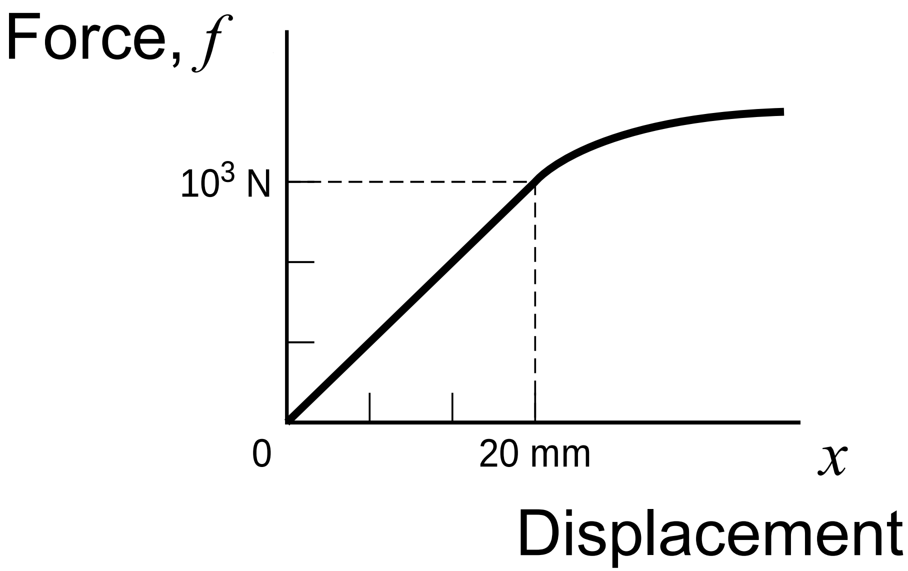
- Vibrations can lead to excessive deflections and failure on the machines and structures
- To reduce vibration through proper design of machines and their mountings
- To utilize profitably in several consumer and industrial applications
- To improve the efficiency of certain machining, casting, forging & welding processes
- To stimulate earthquakes for geological research and conduct studies in design of nuclear reactors

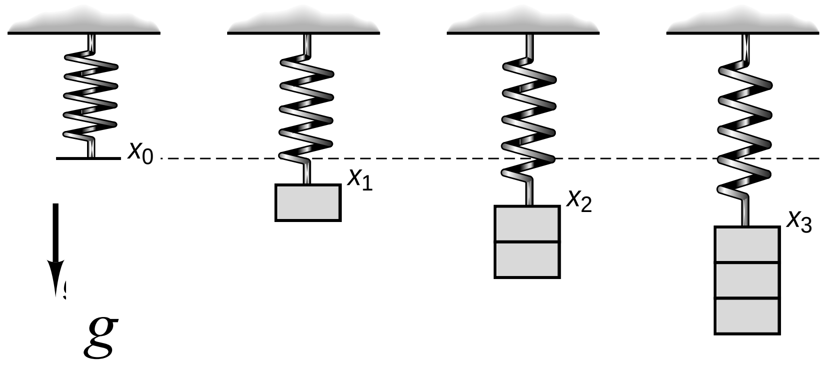
Why to study vibration

- Imbalance in the gas or diesel engines
- Blade and disk vibrations in turbines
- Noise and vibration of the hard-disks in your computers
- Cooling fan in the power supply/computers
- Vibration testing for electronic packaging to conform Internatioal standard for quality control (QC)
- Safety eng.: machine vibration causes parts loose from the body

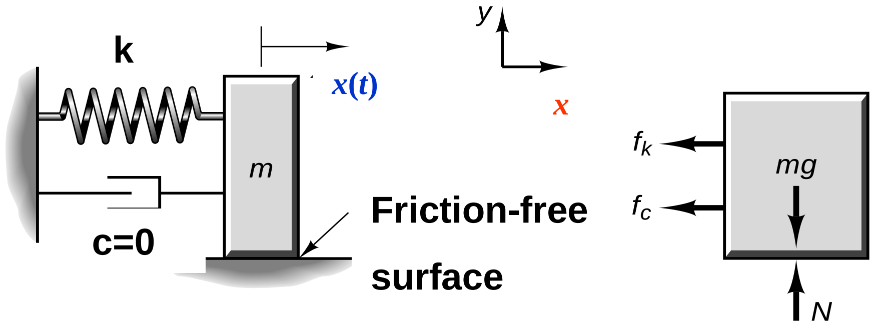
Stiffness

- From strength of materials (Solid Mech) recall:





Free-body diagram and equations of motion



- Newton's Law:

$$m\ddot{x}(t) = -kx(t)$$

$$m\ddot{x}(t) + kx(t) = 0$$

$$x(0) = x_0, \dot{x}(0) = v_0$$

2nd Order Ordinary Differential Equation with Constant Coefficients

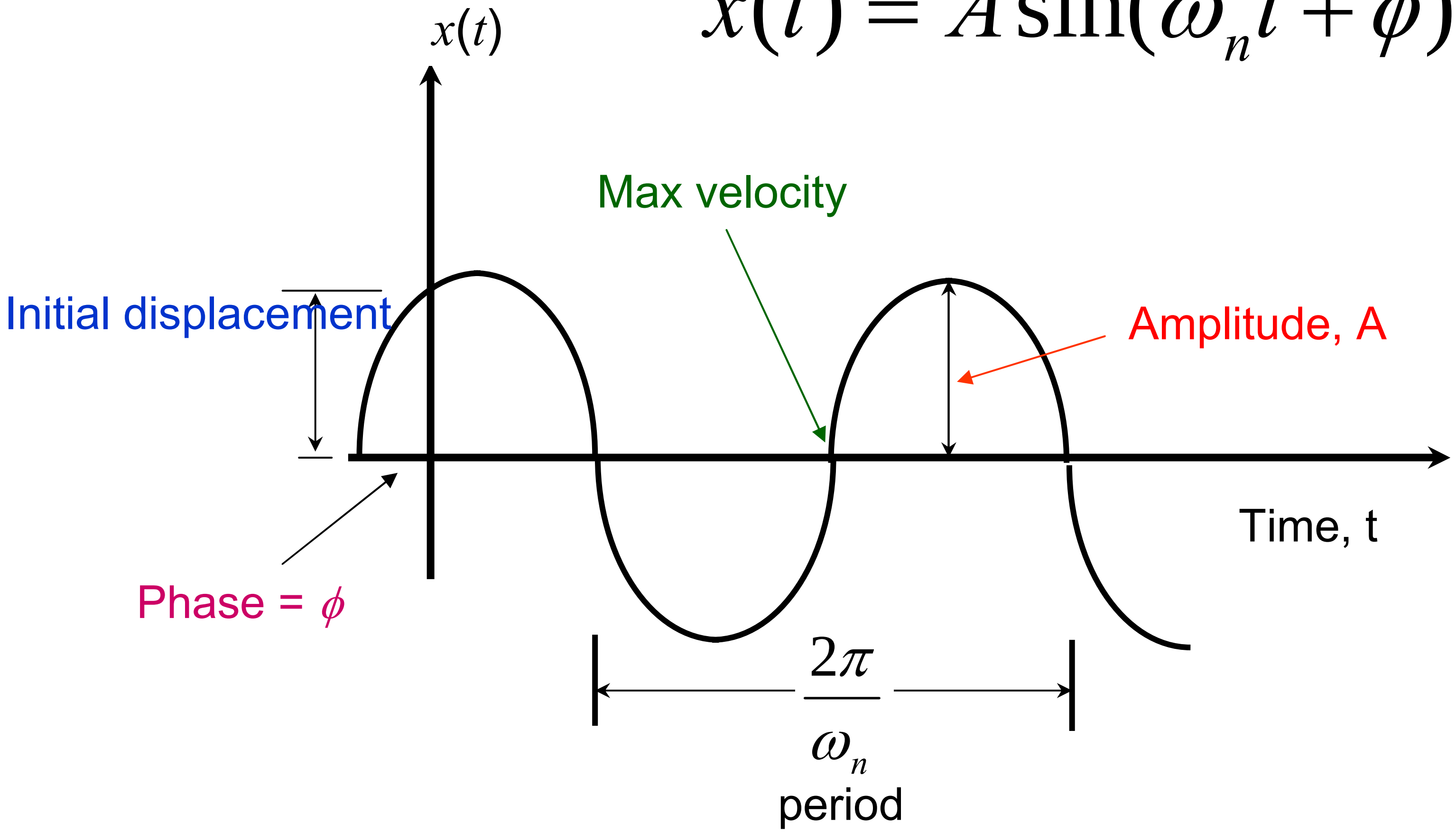
$$\text{Divide by } m : \ddot{x}(t) + \omega_n^2 x(t) = 0$$

$$\omega_n = \sqrt{\frac{k}{m}} \;:\; \text{natural frequency in rad/s}$$

$$x(t) = A\sin(\omega_n t + \phi)$$

Periodic Motion

$$x(t) = A\sin(\omega_n t + \phi)$$



Frequency

ω_n is in rad/s is the natural frequency

$$f_n = \frac{\omega_n \text{ rad/s}}{2\pi \text{ rad/cycle}} = \frac{\omega_n \text{ cycles}}{2\pi \text{ s}} = \frac{\omega_n}{2\pi} \text{Hz}$$

$$T = \frac{2\pi}{\omega_n} \text{ s} \quad \text{is the period}$$

We often speak of frequency in Hertz, but we need rad/s in the arguments of the trigonometric functions (sin and cos function).

Amplitude & Phase from the initial conditions

$$x_0 = A\sin(\omega_n 0 + \phi) = A\sin\phi$$

$$v_0 = \omega_n A\cos(\omega_n 0 + \phi) = \omega_n A\cos\phi$$

Solving yields

$$\underbrace{A = \frac{1}{\omega_n}\sqrt{\omega_n^2 x_0^2 + v_0^2}}_{\text{Amplitude}}, \quad \underbrace{\phi = \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right)}_{\text{Phase}}$$

Phase Relationship between *x*, *v*, *a*

Displacement

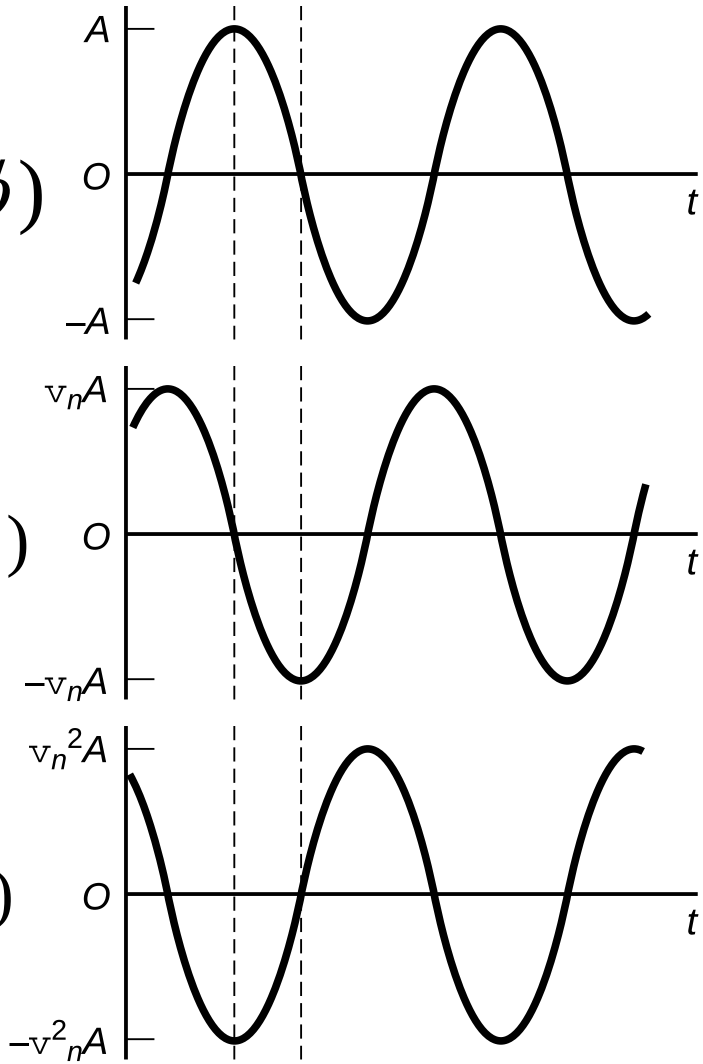
$$x = A\cos(\omega_n t + \phi)$$

Velocity

$$\dot{x} = -\omega_n A\sin(\omega_n t + \phi)$$

Acceleration

$$\ddot{x} = -\omega_n^2 A\cos(\omega_n t + \phi)$$

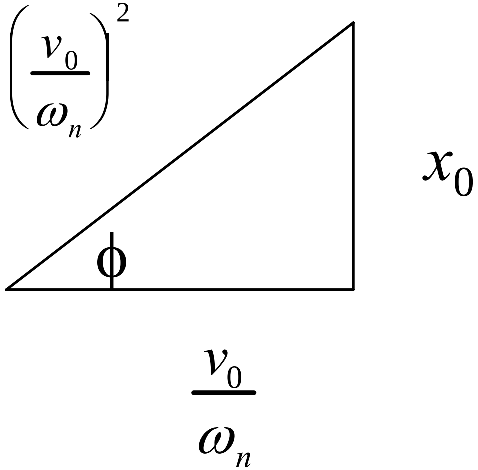


Example 1 verify that equation which satisfies the initial conditions

$$x(t) = A\sin(\omega t + \phi) \Rightarrow x(0) = A\sin(\phi)$$

$$= \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2} \bullet \frac{x_0}{\sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}}$$

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}$$



Example 2 For m= 300 kg and ω_n =10 rad/s compute the stiffness:

$$\omega_n = \sqrt{\frac{k}{m}} \Rightarrow k = m\omega_n^2$$

$$= (300)10^2 \ \mathrm{kg/s}^2$$

$$= 3 \times 10^4 \ \mathrm{N/m}$$

Other forms of the solution:

$$x(t) = A\sin(\omega_n t + \phi)$$

$$x(t) = A_1 \sin\omega_n t + A_2 \cos\omega_n t$$

$$x(t) = a_1 e^{j\omega_n t} + a_2 e^{-j\omega_n t}$$

Phasor: representation of a complex number in terms of a complex exponential $\vec{X} = A(\cos\theta + i\sin\theta) = \underline{Ae^{i\theta}}$

Ref: 1) Sec 1.10.2, 1.10.3

2) http://mathworld.wolfram.com/Phasor.html

Some useful quantities

$$A = \text{ peak value}$$

$$\bar{x} = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t)dt \; = \; \text{average value}$$

$$\bar{x}^2 = \lim_{T\to\infty} \frac{1}{T} \int_0^T x^2(t)dt \; = \; \text{mean - square value}$$

$$x_{rms} = \sqrt{\bar{x}^2} \; = \; \text{root mean square value}$$

Peak Values

$$\text{max or peak value of :}$$

$$\text{displacement : } x_{\max} = A$$

$$\text{velocity : } \dot{x}_{\max} = \omega A$$

$$\text{acceleration : } \ddot{x}_{\max} = \omega^2 A$$

Example 3 Hardware store spring, bolt: m= 49.2x10^{-3} kg,k=857.8 N/m and x_0 =10 mm. Compute ω_n and max amplitude of vibration.

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{857.8 \text{ N/m}}{49.2 \times 10^{-3} \text{ kg}}} = 132 \text{ rad/s}$$

$$f_n = \frac{\omega_n}{2\pi} = 21 \text{ Hz}$$

$$T = \frac{2\pi}{\omega_n} = \frac{1}{f_n} = \frac{1}{21 \, cyles/\text{sec}} 0.0476 \text{ s}$$

$$x(t)_{\max} = A = \frac{1}{\omega_n}\sqrt{\omega_n^2 x_0^2 + \cancel{v_0^2}} = x_0 = 10 \text{ mm}$$

Compute the solution and max velocity and acceleration

$$v(t)_{\max} = \omega_n A = 1320 \text{ mm/s} = 1.32 \text{ m/s}$$

$$a(t)_{\max} = \omega_n^2 A = 174.24 \times 10^3 \text{ mm/s}^2$$

$$= 174.24 \text{ m/s}^2 \approx 17.8g!$$

$$\phi = \tan^{-1}\left(\frac{\omega_n x_0}{0}\right) = \frac{\pi}{2} \text{ rad}$$

$$x(t) = 10\sin(132t + \pi/2) = 10\cos(132t) \text{ mm}$$

A note on arctangents

- Note that using the arctangent from a machine requires some attention
- The argument atan(-/+) is in a different quadrant then atan(+/-), and usual machine calculations will return an arctangent in between $-\pi/2$ and $+\pi/2$ reading only the atan(-) for both of the above two cases.
- In Matlab: atan(z) and atan2(y,x)

Derivation of the solution

Substitute $x(t) = ae^{\lambda t}$ into $m\ddot{x} + kx = 0 \Rightarrow$

$$m\lambda^2 ae^{\lambda t} + kae^{\lambda t} = 0 \Rightarrow$$

$$m\lambda^2 + k = 0 \Rightarrow$$

$$\lambda = \pm\sqrt{-\frac{k}{m}} = \pm\sqrt{\frac{k}{m}}j = \pm\omega_n j \Rightarrow$$

$$x(t) = a_1 e^{\omega_n jt} \text{ and } x(t) = a_2 e^{-\omega_n jt} \Rightarrow$$

$$x(t) = a_1 e^{\omega_n jt} + a_2 e^{-\omega_n jt}$$

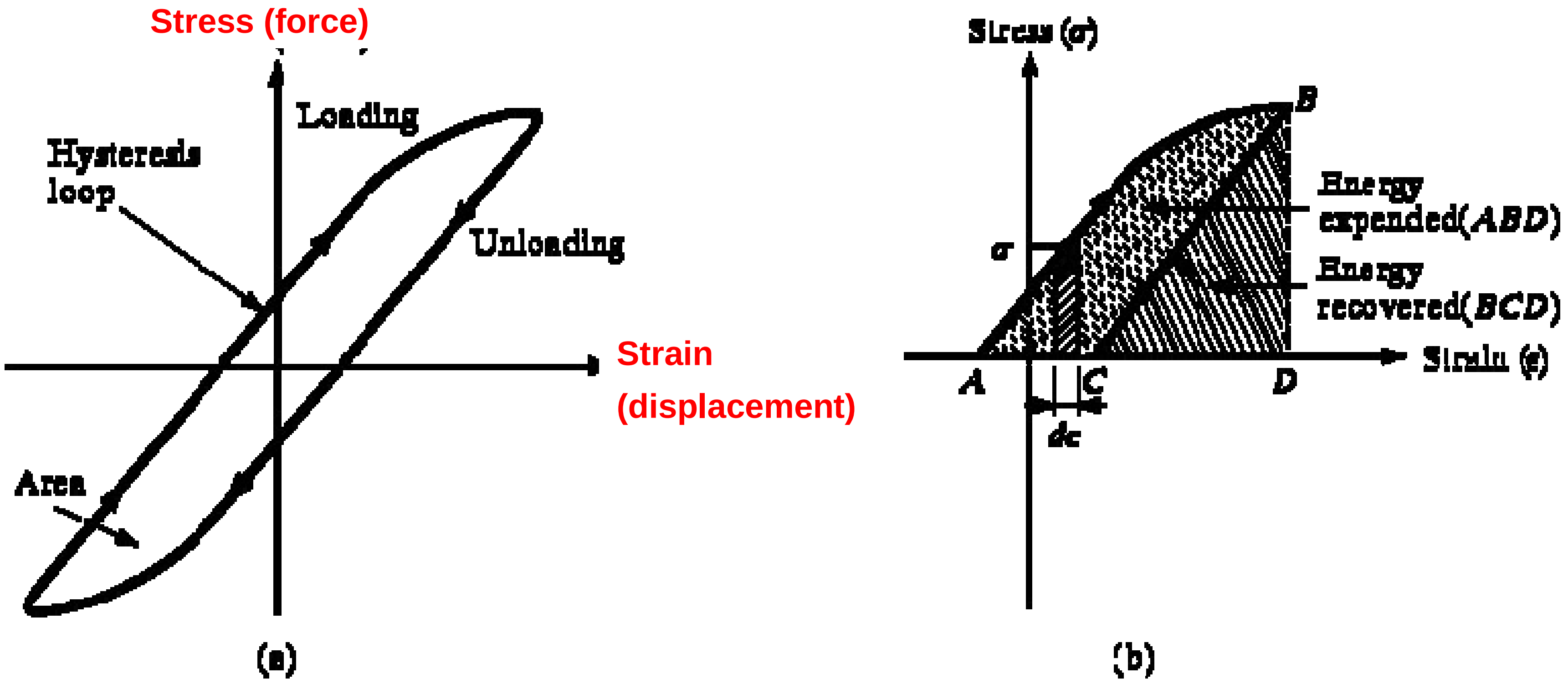
Damping Elements

- Viscous Damping:
Damping force is proportional to the velocity of the vibrating body in a fluid medium such as air, water, gas, and oil.

- Coulomb or Dry Friction Damping:
Damping force is constant in magnitude but opposite in direction to that of the motion of the vibrating body between dry surfaces

- Material or Solid or Hysteretic Damping:
Energy is absorbed or dissipated by material during deformation due to friction between internal planes

Hysteresis loop for elastic materials



Viscous Damping

❑ *Shear Stress (τ)* developed in the fluid layer at a distance y from the fixed plate is:

$$\tau = \mu \frac{du}{dy} \qquad (1.26)$$

where $du/dy = v/h$ is the velocity gradient.

• *Shear* or *Resisting Force (F)* developed at the bottom surface of the moving plate is:

$$F = \tau A = \mu \frac{Av}{h} = cv \qquad (1.27)$$

where A is the surface area of the moving plate.

$\boxed{c = \frac{\mu A}{h}}$ is the damping constant

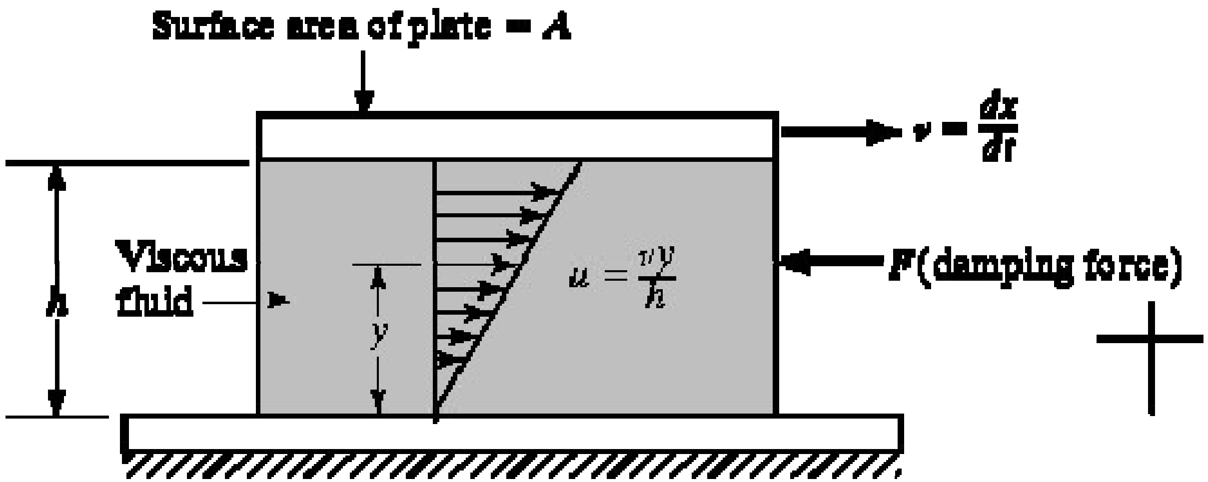
Viscous Damping

and $$c = \frac{\mu A}{h} \qquad (1.28)$$

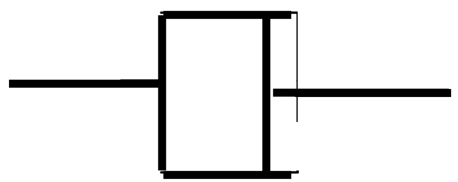
is called the damping constant.

- If a damper is nonlinear, a linearization process is used about the operating velocity (v*) and the equivalent damping constant is:

$$c = \left.\frac{dF}{dv}\right|_{v^*} \qquad (1.29)$$



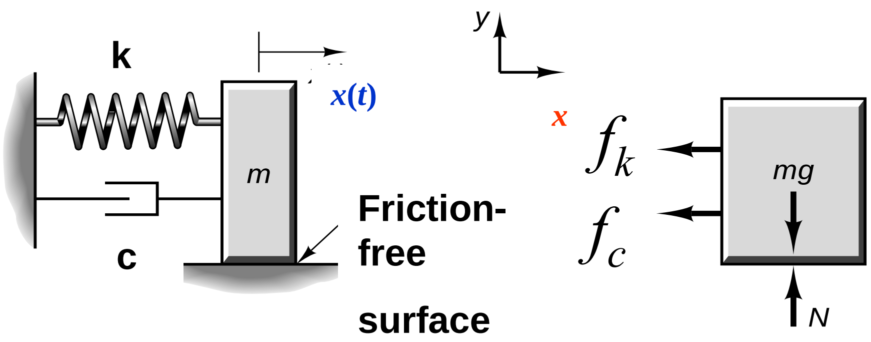
Linear Viscous Damping



$$f_c = c\dot{x}(t)$$

- A mathematical form
- Called a dashpot or viscous damper
- Somewhat like a shock absorber
- The constant c has units: Ns/m or kg/s

Spring-mass-damper systems



- From Newton's law:

$$m\ddot{x}(t) = -f_c - f_k$$
$$= -c\dot{x}(t) - kx(t)$$
$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0$$
$$x(0) = x_0, \ \dot{x}(0) = v_0$$

Derivation of the solution

$$\text{Substitute} \quad x(t) = ae^{\lambda t} \quad \text{into} \quad m\ddot{x} + c\dot{x} + kx = 0 \Rightarrow$$

$$m\lambda^2 ae^{\lambda t} + c\lambda ae^{\lambda t} + kae^{\lambda t} = 0 \Rightarrow$$

$$m\lambda^2 + c\lambda + k = 0 \Rightarrow$$

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \Rightarrow$$

$$x(t) = a_1 e^{\lambda_1 t} \quad \text{and} \quad x(t) = a_2 e^{\lambda_2 t} \Rightarrow$$

$$x(t) = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t}$$

Solution of SDOF M-C-K System
(dates to 1743 by Euler)

Divide equation of motion by m

$$\ddot{x}(t)+2\zeta\omega_n\dot{x}(t)+\omega_n^2 x(t)=0$$

where $\omega_n=\sqrt{k/m}$ and

$$\zeta=\frac{c}{2\sqrt{km}}=\frac{c}{c_{cr}}=\text{damping ratio (dimensionless)}$$

$$c_{cr}=2\sqrt{km}$$

Solution of SDOF M-C-K System

Let $x(t) = ae^{\lambda t}$ & subsitute into eq. of motion

$$\lambda^2 ae^{\lambda t} + 2\zeta\omega_n \lambda ae^{\lambda t} + \omega_n^2 ae^{\lambda t} = 0$$

which is now an algebraic equation in λ:

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

from the roots of a quadratic equation

Here the discriminant $\zeta^2 - 1$, determines

the nature of the roots λ

Three possibilities:

1) $\zeta = 1 \Rightarrow$ roots are equal & repeated

called critically damped

$$\zeta = 1 \Rightarrow c = c_{cr} = 2\sqrt{km} = 2m\omega_n$$

$$x(t) = a_1 e^{-\omega_n t} + a_2 t e^{-\omega_n t}$$

Using the initial conditions:

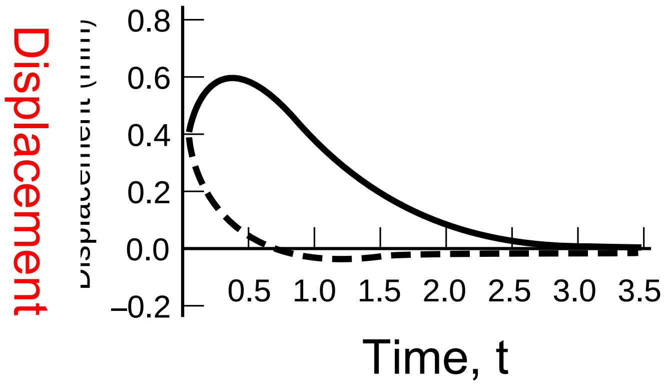
$$a_1 = x_0, \; a_2 = v_0 + \omega_n x_0$$

Sec. 2.6

Critical damping continued

- No oscillation occurs
- Useful in door mechanisms, analog gauges

$$x(t) = [x_0 + (v_0 + \omega_n x_0)t]e^{-\omega_n t}$$



http://www.amazon.com/s?ie=UTF8&rh=n%3A3407141&page=1

Overdamping

2) $\zeta > 1$, called overdamping - two distinct real roots:

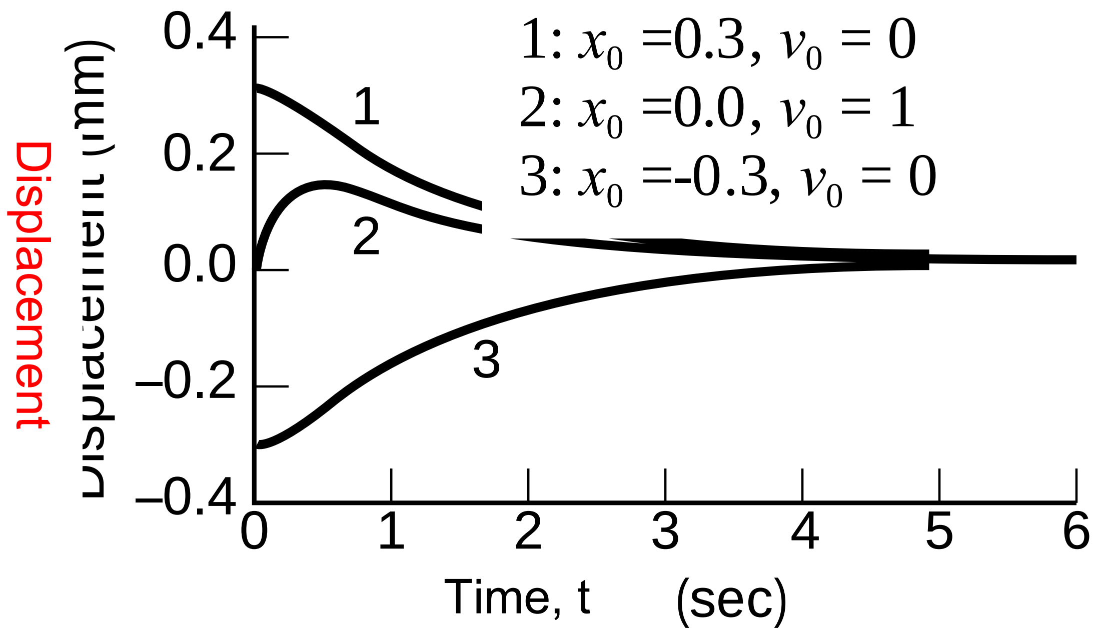
$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$x(t) = e^{-\zeta\omega_n t}(a_1 e^{-\omega_n t\sqrt{\zeta^2-1}} + a_2 e^{\omega_n t\sqrt{\zeta^2-1}})$$

where $a_1 = \dfrac{-v_0 + (-\zeta + \sqrt{\zeta^2 - 1})\omega_n x_0}{2\omega_n\sqrt{\zeta^2 - 1}}$

$$a_2 = \frac{v_0 + (\zeta + \sqrt{\zeta^2 - 1})\omega_n x_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

The overdamped response



Under-damping

3) $\zeta < 1,$ called underdamped motion - most common

Two complex roots as conjugate pairs

write roots in complex form as :

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n j\sqrt{1-\zeta^2}$$

where $\quad j = \sqrt{-1}$

Underdamping

$$x(t) = e^{-\zeta\omega_n t}\left(a_1 e^{j\omega_n t\sqrt{1-\zeta^2}} + a_2 e^{-j\omega_n t\sqrt{1-\zeta^2}}\right)$$

$$= A e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

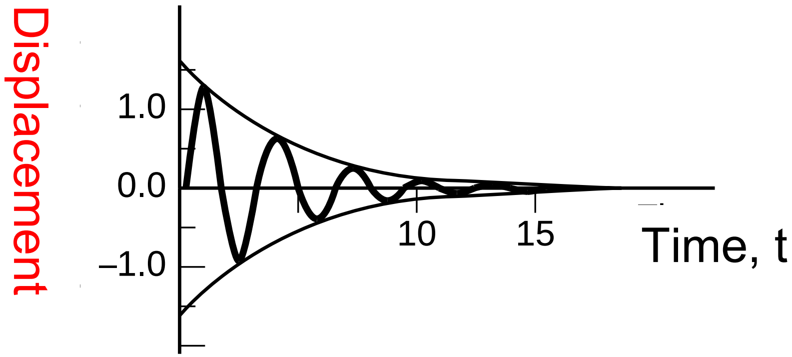
$\omega_d = \omega_n\sqrt{1-\zeta^2}$, damped natural frequency

$$A = \frac{1}{\omega_d}\sqrt{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}$$

$$\phi = \tan^{-1}\left(\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right)$$

http://acoustics.me.uic.edu/Simulation/SDOF%20Damped.htm

Underdamped-oscillation



- Gives an oscillating response with exponential decay
- Most natural systems vibrate with and underdamped response
- See textbook for details and other representations

Example 4 consider the spring in Ex.3, if c = 0.11 kg/s,

determine the damping ratio of the spring-bolt system.

$$m = 49.2 \times 10^{-3}\ \text{kg},\quad k = 857.8\ \text{N/m}$$

$$c_{cr} = 2\sqrt{km} = 2\sqrt{49.2 \times 10^{-3} \times 857.8}$$

$$= 12.993\ \text{kg/s}$$

$$\zeta = \frac{c}{c_{cr}} = \frac{0.11\ \text{kg/s}}{12.993\ \text{kg/s}} = 0.0085 \Rightarrow$$

the motion is *underdamped*

and the bolt will oscillate

Example 5

The human leg has a measured natural frequency of around 20 Hz (Ref: Fig. 9.2) when in its rigid (knee locked) position, in the longitudinal direction (i.e., along the length of the bone) with a damping ratio of $\zeta = 0.224$.

Calculate the response of the tip if the leg bone to $v_0 = 0.6$ m/s and $x_0 = 0$

This correspond to the vibration induced while landing on your feet, with your knees locked from a height of 18 mm) and plot the response. What is the maximum acceleration experienced by the leg assuming no damping?

Solution:

V_0=0.6, X_0=0, ζ = 0.224

$$\omega_n = \frac{20}{1}\frac{\text{cycles}}{s}\frac{2\pi\,\text{rad}}{\text{cycles}} = 125.66 \text{ rad/s}$$

$$\omega_d = 125.66\sqrt{1-(.224)^2} = 122.467 \text{ rad/s}$$

$$A = \frac{\sqrt{(0.6+(0.224)(125.66)(0))^2 + (0)(122.467)^2}}{122.467} = 0.005 \text{ m}$$

$$\boxed{A = \frac{1}{\omega_d}\sqrt{(v_0+\zeta\omega_n x_0)^2 + (x_0\omega_d)^2}}$$

$$\phi = \tan^{-1}\left(\frac{(0)(\omega_d)}{v_0+\zeta\omega_n(0)}\right) = 0$$

$$\boxed{\phi = \tan^{-1}\left(\frac{x_0\omega_d}{v_0+\zeta\omega_n x_0}\right)}$$

$$\Rightarrow \underline{x(t) = 0.005e^{-28.148t}\sin(122.467t)}$$

Use <u>undamped formula</u> to get max acceleration:

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}, \omega_n = 125.66, v_0 = 0.6, x_0 = 0$$

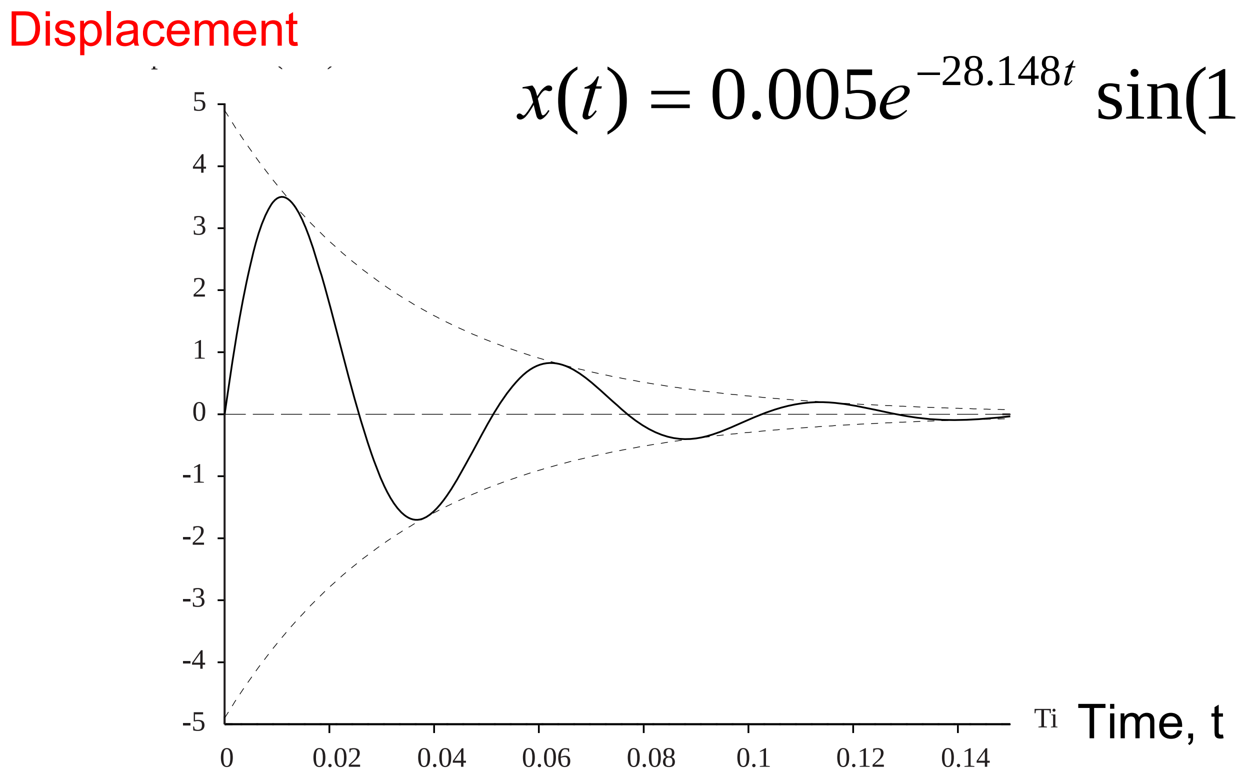
$$A = \frac{v_0}{\omega_n}\,\mathrm{m} = \frac{0.6}{\omega_n}\,\mathrm{m}$$

$$\max(\ddot{x}) = \left|-\omega_n^2 A\right| = \left|-\omega_n^2\left(\frac{0.6}{\omega_n}\right)\right| = (0.6)\left(125.66\ \mathrm{m/s^2}\right) = \underline{75.396\ \mathrm{m/s^2}}$$

$$\text{maximum acceleration} = \frac{75.396\ \mathrm{m/s^2}}{9.81\ \mathrm{m/s^2}}\,g = \underline{7.68\,\mathrm{g's}}$$

Plot of the response:

$$x(t) = 0.005e^{-28.148t}\sin(122.467t)$$



Example 6 Compute the form of the response of an underdamped system using the Cartesian form

$$\sin(x+y)=\sin x\cos y-\cos x\sin y \Rightarrow$$

$$x(t)=Ae^{-\zeta\omega_n t}\sin(\omega_d t+\phi)=e^{-\zeta\omega_n t}(A_1\sin\omega_d t+A_2\cos\omega_d t)$$

$$x(0)=x_0=e^0(A_1\sin(0)+A_2\cos(0))\Rightarrow \underline{A_2=x_0}$$

$$\dot{x}=-\zeta\omega_n e^{-\zeta\omega_n t}(A_1\sin\omega_d t+A_2\cos\omega_d t)$$
$$+\omega_d e^{-\zeta\omega_n t}(A_1\cos\omega_d t-A_2\sin\omega_d t)$$

$$v_0=-\zeta\omega_n(A_1\sin 0+x_0\cos 0)+\omega_d(A_1\cos 0-x_0\sin 0)$$

$$\Rightarrow \underline{A_1=\frac{v_0+\zeta\omega_n x_0}{\omega_d}}\Rightarrow$$

$$x(t)=e^{-\zeta\omega_n t}\left(\frac{v_0+\zeta\omega_n x_0}{\omega_d}\sin\omega_d t+x_0\cos\omega_d t\right)$$

$\boxed{\text{Eq. 2.72}}$

MODELING AND ENERGY METHODS

An alternative way to determine the equation of motion and an alternative way to calculate the natural frequency

Modelling

- Newton's Laws

$$\sum F_{xi} = m\ddot{x}$$

$$\sum M_{0i} = I_0\ddot{\theta}$$

Energy Methods

$$\int F dx = \int m\ddot{x} dx \Rightarrow$$

$$\text{work done} = \overbrace{U_1 - U_2}^{\text{Potential Energy}} = \frac{1}{2} m\dot{x}^2 \Big|_1^2 = \underbrace{T_2 - T_1}_{\text{Kinetic Energy}}$$

$$\Rightarrow T + U = \text{constant}$$

or $$\frac{d}{dt}(T + U) = 0$$

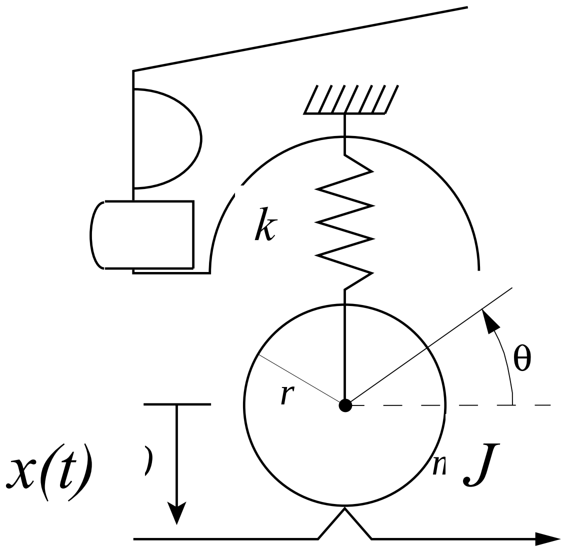
Alternate method of getting the eq. of motion

Rayleigh's Method

- $T_1 + U_1 = T_2 + U_2$
- Let t_1 be the time at which m moves through its static equilibrium position, then
- $U_1=0$, reference point
- Let t_2 be the time at which m undergoes its max displacement ($v=0$ so $T_2=0$), U_2 is max (T_1 must be max),
- Thus $U_{max}=T_{max}$

Ref: Section 2.5

Example 7



Compute the natural frequency of this simplified model of a car hitting a bump. Assume it is a conservative system.

$$T_{\text{rotational}} = \frac{1}{2} J\dot{\theta}^2 \quad \text{and} \quad T_{\text{Translational}} = \frac{1}{2} m\dot{x}^2$$

J **= mass moment of inertia (rotational mass, rotational inertia, angular mass) = m r²**

Solution continued

$$x = r\theta \Rightarrow \dot{x} = r\dot{\theta} \Rightarrow T_{\text{Rot}} = \frac{1}{2} J \frac{\dot{x}^2}{r^2}$$

The max value of T happens at $v_{\max} = \dot{x}_{\max} = \omega_n A$

$$\Rightarrow T_{\max} = \frac{1}{2} J \frac{(\omega_n A)^2}{r^2} + \frac{1}{2} m(\omega_n A)^2 = \frac{1}{2}\left(m + \frac{J}{r^2}\right)\omega_n^2 A^2$$

The max value of U happens at $x_{\max} = A$

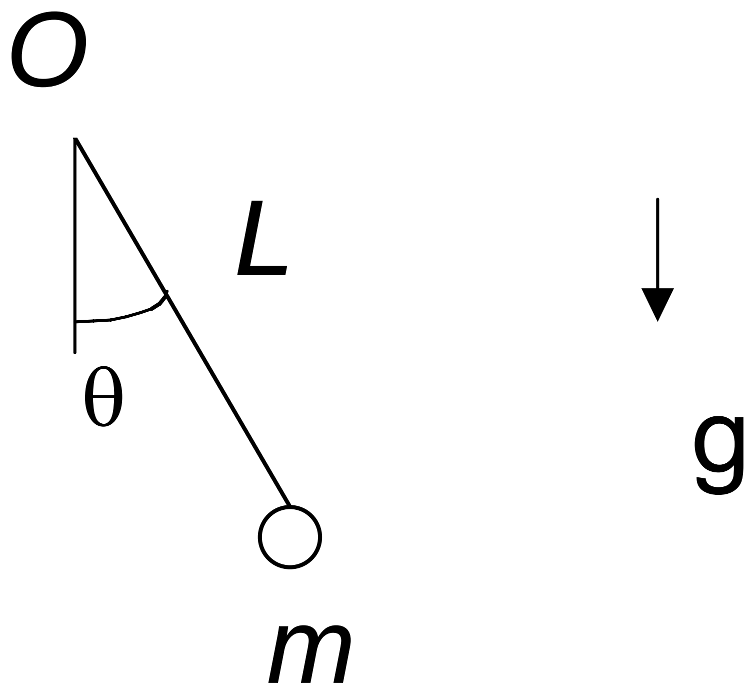
$$\Rightarrow U_{\max} = \frac{1}{2} kA^2$$

Thus $T_{\max} = U_{\max} \Rightarrow$

$$\frac{1}{2}\left(m + \frac{J}{r^2}\right)\omega_n^2 A^2 = \frac{1}{2} kA^2 \Rightarrow \omega_n = \sqrt{\frac{k}{\boxed{m + \frac{J}{r^2}}}}$$

Effective mass
(Total effective mass)

Example 8: Pendulum



$J_0 = mL^2$

Now write down the energy

$$T = \frac{1}{2} J_0 \dot{\theta}^2 = \frac{1}{2} m\ell^2 \dot{\theta}^2$$

$$U = mg\ell(1-\cos\theta),\ \text{the change in elevation}$$
$$\text{is } \ell(1-\cos\theta)$$

$$\frac{d}{dt}(T+U) = \frac{d}{dt}\left(\frac{1}{2} m\ell^2 \dot{\theta}^2 + mg\ell(1-\cos\theta)\right) = 0$$

$$m\ell^2 \underline{\dot{\theta}}\ddot{\theta} + mg\ell(\sin\theta)\underline{\dot{\theta}} = 0$$

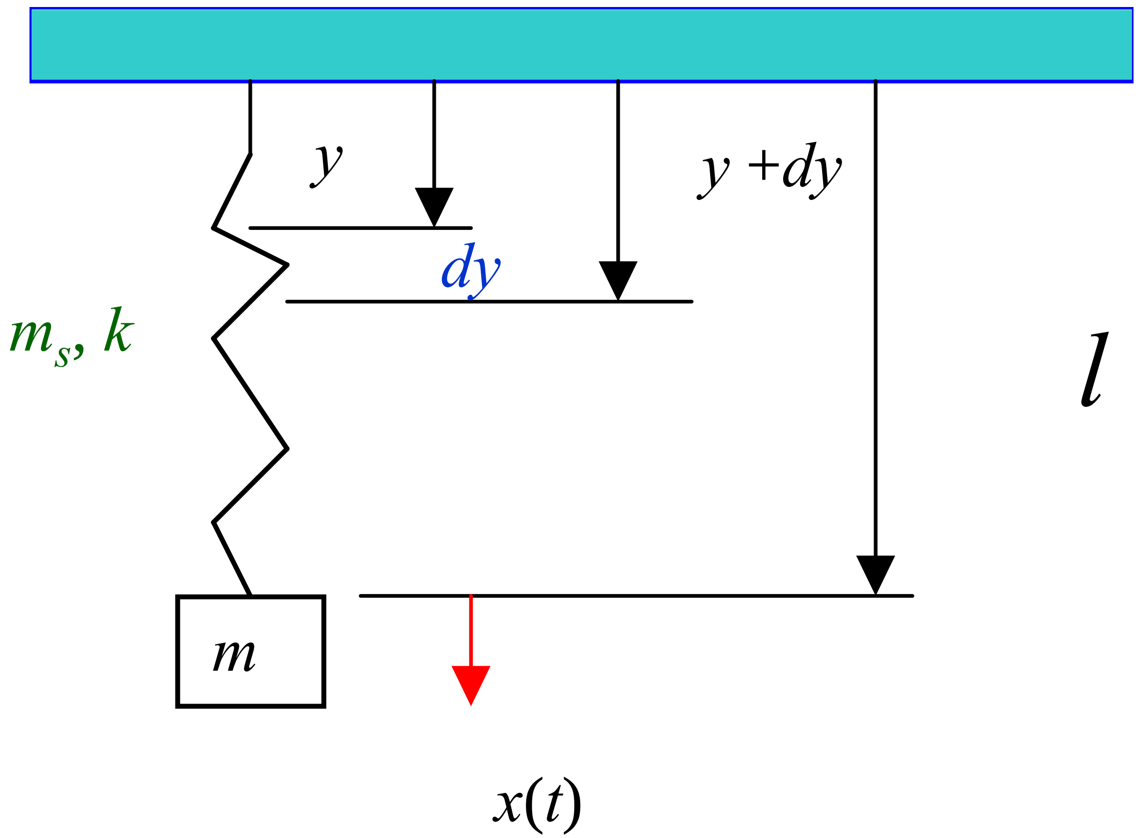
$$\Rightarrow \dot{\theta}\left(m\ell^2\ddot{\theta} + mg\ell(\sin\theta)\right) = 0$$

$$\Rightarrow m\ell^2\ddot{\theta} + mg\ell(\sin\theta) = 0$$

$$\Rightarrow \ddot{\theta}(t) + \frac{g}{\ell}\sin\theta(t) = 0$$ for small θ, sin θ~ θ

$$\Rightarrow \ddot{\theta}(t) + \frac{g}{\ell}\theta(t) = 0 \quad \Rightarrow \omega_n = \sqrt{\frac{g}{\ell}}$$

Example 9 The effect of including the mass of the spring on the value of the frequency.



Ex. 2.8

Solution to Ex 9

$$\left.\begin{aligned}&\text{mass of element } dy: \frac{m_s}{\ell}dy\\ &\text{velocity of element } dy: v_{dy} = \frac{y}{\ell}\dot{x}(t),\end{aligned}\right\}\text{assumptions}$$

$$T_{spring} = \frac{1}{2}\int_0^\ell \frac{m_s}{\ell}\left[\frac{y}{\ell}\dot{x}\right]^2 dy \quad \text{(adds up the KE of each differential element)}$$

$$= \frac{1}{2}\left(\frac{m_s}{3}\right)\dot{x}^2$$

$$T_{mass} = \frac{1}{2}m\dot{x}^2 \Rightarrow T_{tot} = \left[\frac{1}{2}\left(\frac{m_s}{3}\right) + \frac{1}{2}m\right]\dot{x}^2 \Rightarrow T_{\max} = \frac{1}{2}\left(m + \frac{m_s}{3}\right)\omega_n^2 A_n^2$$

$$U_{\max} = \frac{1}{2}kA^2$$

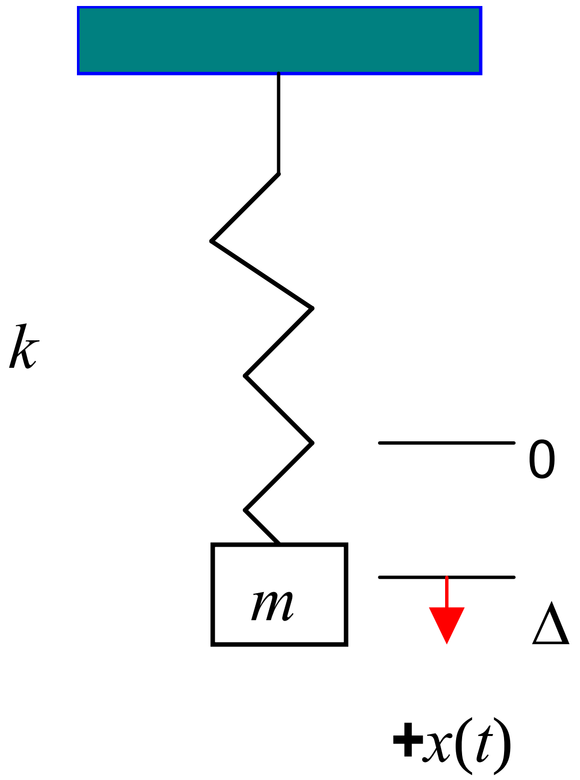
$$\boxed{U_{\max} = T_{\max}} \qquad \Rightarrow \omega_n = \sqrt{\frac{k}{\boxed{m + \frac{m_s}{3}}}}$$

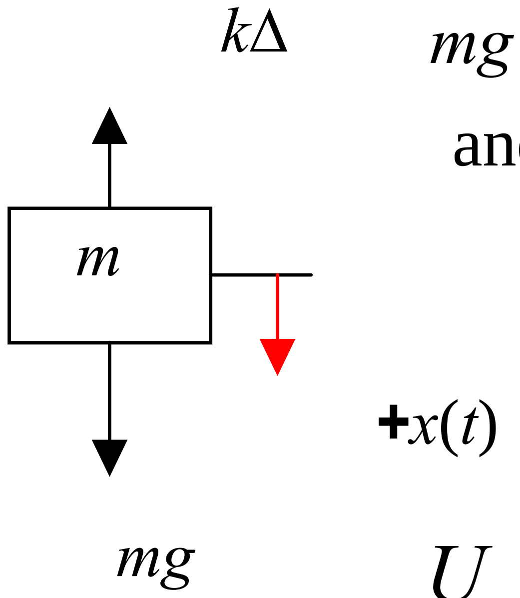
Provides some simple design and modeling guides

Effect of the spring mass = add 1/3 of its mass to the main mass

Ex. 2.8

What about gravity?





$mg - k\Delta = 0$, from FBD, and static equilibrium

$$U_{spring} = \frac{1}{2}k(\Delta + x)^2$$

$$U_{grav} = -mgx$$

$$T = \frac{1}{2}m\dot{x}^2$$

$$\text{Now use } \frac{d}{dt}(T+U)=0$$

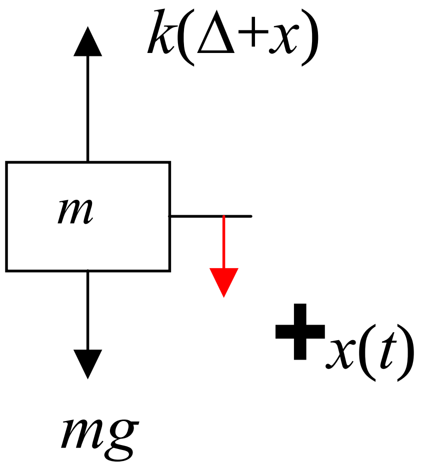
$$\Rightarrow \frac{d}{dt}\left[\frac{1}{2}m\dot{x}^2 - mgx + \frac{1}{2}k(\Delta + x)^2\right] = 0$$

$$\Rightarrow m\dot{x}\ddot{x} - mg\dot{x} + k(\Delta + x)\dot{x}$$

$$\Rightarrow \dot{x}(m\ddot{x} + kx) + \dot{x}(\underbrace{k\Delta - mg}_{\text{0 from static equ.}}) = 0$$

$$\Rightarrow m\ddot{x} + kx = 0$$

Do it again using Newton's law

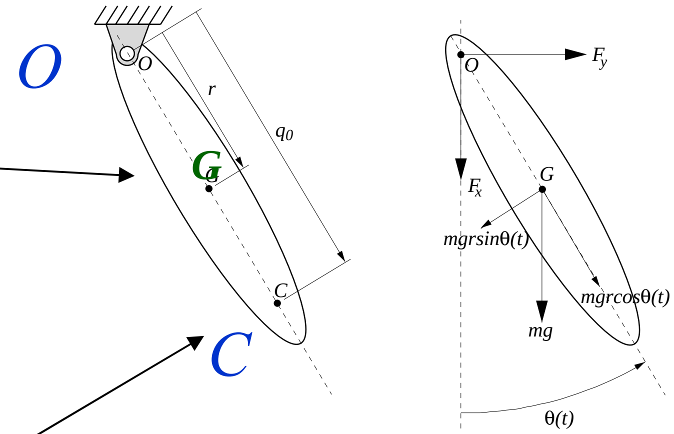


$$\text{From FBD of mass:}$$

$$m\ddot{x} = mg - k(\Delta + x) = -kx + \underbrace{(mg - k\Delta)}_{\substack{\text{0 from static} \\ \text{equilibrium}}}$$

$$\Rightarrow \underline{m\ddot{x} + kx = 0}$$

Example Compound Pendulum



G is the center of mass

r is the distance *OG*

C is the center of percussion
Defined as the distance

$$q_0 = \frac{J}{mr}$$

q_0=Length of equivalent simple pendulum

where a simple pendulum of the same mass would have the same period (T) as this pendulum

The radius of gyration k_0 is the radius of a ring that would have the same angular inertia

$$k_0 = \sqrt{q_0 r}$$

Ex. 2.6

$$\sum M_0 = J\ddot{\theta}(t) = -mgr\sin\theta(t)$$

$$\Rightarrow J\ddot{\theta}(t) + mgr\sin\theta(t) = 0$$

$$\Rightarrow \ddot{\theta}(t) + \frac{mgr}{J}\sin\theta(t) = 0$$

$$\sin\theta \approx \theta \Rightarrow \ddot{\theta}(t) + \frac{mgr}{J}\theta(t) = 0$$

$$\Rightarrow \omega_n = \sqrt{\frac{mgr}{J}} = \sqrt{\frac{g}{q_0}}$$

center of percussion = the point on a rigid body, suspended so as to be able to move freely about a fixed axis, at which the body may be struck without changing the position of the axis

No matter whether the body is pivoted from O or C, its natural frequency is the same. The point C is called the center of percussion.

Center of Percussion in ASTM Standard for quality control

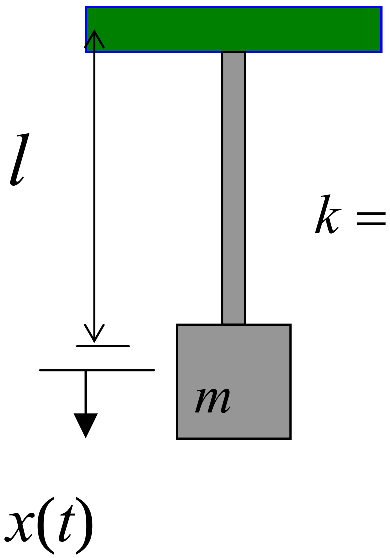
The method used by manufacturers and governing associations to determine the moment-of-inertia of a baseball or softball bat is to measure the period of oscillation when the bat is allowed to swing as a pendulum from a pivot point 6-inches from the knob end of the bat, and then calculate the moment-of-inertia

ASTM F 1881-04: Standard Test Method for Measuring Baseball Bat Performance Factor, and ASTM F 1890-04: Standard Test Method for Measuring Softball Performance Factor, Annual Book of ASTM Standards, Vol. 15.07 (ASTM International, West Conshohocken, PA, 2004). ASTM F 2398: Test Method for Measuring Moment of Inertia and Center of Percussion of a Baseball or Softball Bat

ASTM= American Society for Testing and Materials International

www.astm.org

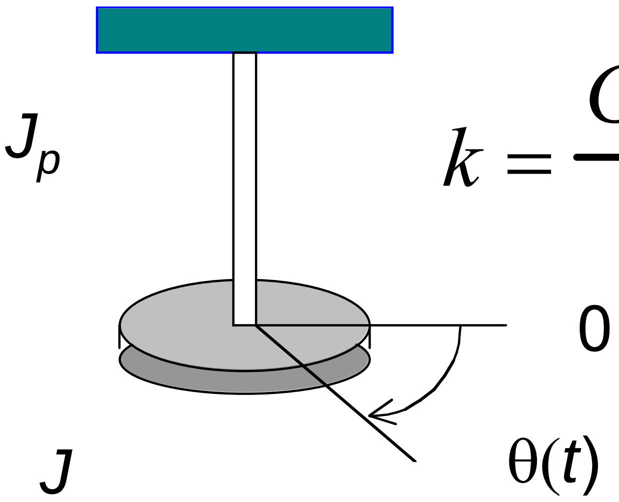
More on springs and stiffness



$$k = \frac{EA}{\ell}$$

- Longitudinal motion
- A is the cross sectional area (m^2)
- E is the elastic modulus ($Pa=N/m^2$)
- l is the length (m)
- k is the stiffness (N/m)

Torsional Stiffness



$$k = \frac{GJ_p}{\ell}$$

- J_p is the polar moment of inertia of the rod
- J is the mass moment of inertia of the disk
- G is the shear modulus, l is the length

$$J_p = \int r^2 dA \qquad J = \int r^2 dm$$

Polar moment of inertia (Jp) is a measure of an object's ability to resist torsion (solid mechics). The polar moment of inertia must not be confused with the moment of inertia (J), which characterizes an object's angular acceleration due to a torque (dynamics).

Sec. 2.3

Example compute the frequency of a shaft/mass system {J = 0.5 kg m^2}

$$\sum M = J\ddot{\theta} \Rightarrow J\ddot{\theta}(t) + k\theta(t) = 0$$

$$\Rightarrow \ddot{\theta}(t) + \frac{k}{J}\theta(t) = 0$$

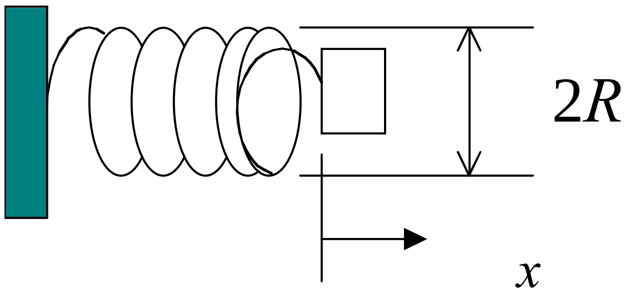
$$\Rightarrow \omega_n = \sqrt{\frac{k}{J}} = \sqrt{\frac{GJ_p}{\ell J}}, \quad J_p = \frac{\pi d^4}{32}$$

For a 2 m steel shaft, diameter of 0.5 cm $\Rightarrow$

$$\omega_n = \sqrt{\frac{GJ_p}{\ell J}} = \sqrt{\frac{(8\times 10^{10}\,\mathrm{N/m^2})[\pi(0.5\times 10^{-2}\,\mathrm{m})^4/32]}{(2\,\mathrm{m})(0.5\mathrm{kg}\cdot\mathrm{m^2})}}$$

$$= 2.2\ \mathrm{rad/s}$$

Helical Spring



d = diameter of wire

$2R$= diameter of turns

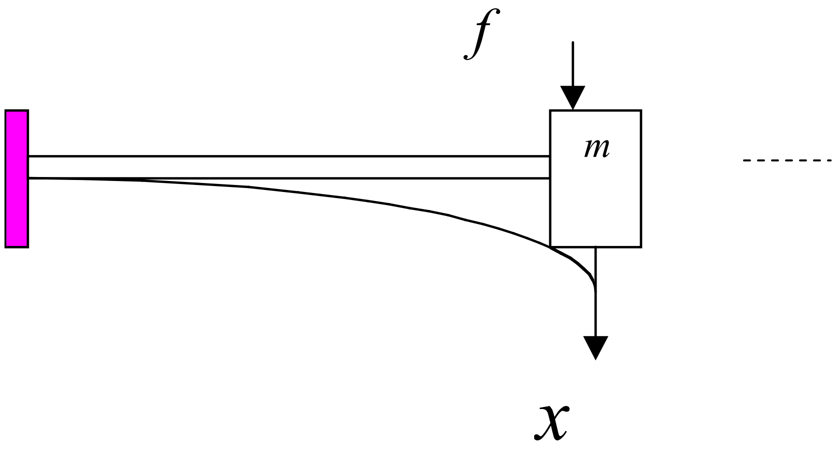
n = number of turns

x= end deflection

G= shear modulus of spring material

$$k = \frac{Gd^4}{64nR^3}$$

Transverse beam stiffness



- Strength of materials and experiments yield:

$$k = \frac{3EI}{\ell^3}$$

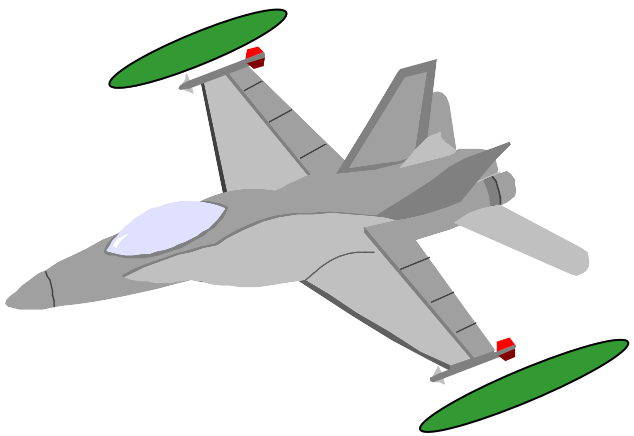
$$\omega_n = \sqrt{\frac{3EI}{m\ell^3}}$$

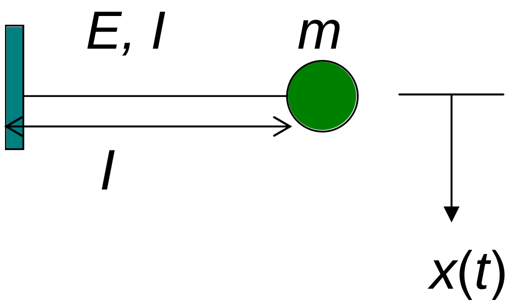
Dr. Y K Lee

Samples of Vibrating Systems

- Deflection of continuum (beams, plates, bars, etc) such as airplane wings, truck chassis, disc drives, circuit boards…
- Shaft rotation
- Rolling ships
- See text for more examples.

Example : effect of fuel on frequency of an airplane wing





- Model wing as transverse beam
- Model fuel as tip mass
- Ignore the mass of the wing and see how the frequency of the system changes as the fuel is used up

Mass of pod 10 kg empty 1000 kg full
I = 5.2x10^{-5} m^4, E =6.9x10^9 N/m, l = 2 m

- Hence the natural frequency changes by an order of magnitude while it empties out fuel.

$$\omega_{\text{full}} = \sqrt{\frac{3EI}{m\ell^3}} = \sqrt{\frac{3(6.9\times10^9)(5.2\times10^{-5})}{1000\cdot 2^3}}$$

$$= 11.6 \text{ rad/s} = 1.8 \text{ Hz}$$

$$\omega_{\text{empty}} = \sqrt{\frac{3EI}{m\ell^3}} = \sqrt{\frac{3(6.9\times10^9)(5.2\times10^{-5})}{10\cdot 2^3}}$$

$$= 115 \text{ rad/s} = 18.5 \text{ Hz}$$

Pod= a streamlined external housing that enclose engines or fuel

Static Deflection

$$\delta, \Delta = \text{distance spring is stretched or compressed under the force of gravity by attaching a mass } m \text{ to it.}$$

$$\Delta = \delta = \delta_s = \frac{mg}{k}$$

Many symbols in use including x_s and x_0

Combining Springs

- Equivalent Spring

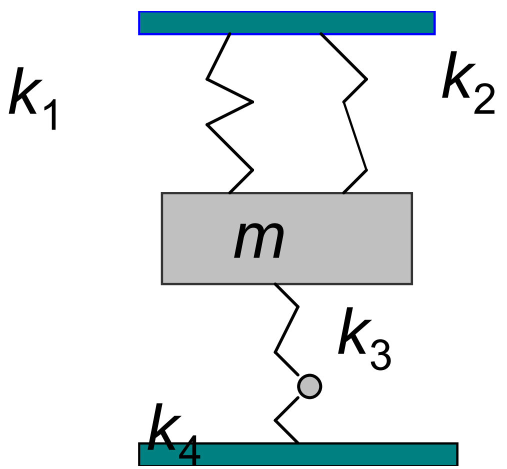
$$\text{series}: \ k_{AC} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}}$$

$$\text{parallel}: \ k_{ab} = k_1 + k_2$$

Use these to design from available parts

- Discrete springs available in standard values
- Dynamic requirements require specific frequencies
- Mass is often fixed or $\pm$ small amount
- Use spring combinations to adjust w_n
- Check static deflection

Example Design of a spring mass system using available springs: series vs parallel



- Let m = 10 kg
- Compare a series and parallel combination
- a) k_1 =1000 N/m, k_2 = 3000 N/m, $k_3 = k_4$ =0
- b) k_3 =1000 N/m, k_4 = 3000 N/m, $k_1 = k_2$ =0

Case a) parallel connection :

$$k_3 = k_4 = 0, k_{eq} = k_1 + k_2 = 1000 + 3000 = 4000 \text{ N/m}$$

$$\Rightarrow \omega_{parallel} = \sqrt{\frac{k_{eg}}{m}} = \sqrt{\frac{4000}{10}} = 20 \text{ rad/s}$$

Case b) series connection :

$$k_1 = k_2 = 0, k_{eq} = \frac{1}{(1/k_3) + (1/k_4)} = \frac{3000}{3+1} = 750 \text{ N/m}$$

$$\Rightarrow \omega_{series} = \sqrt{\frac{k_{eg}}{m}} = \sqrt{\frac{750}{10}} = 8.66 \text{ rad/s}$$

Same physical components, very different frequency
Allows some design flexibility in using off-the-shelf components

Free Vibration with Coulomb Damping

- Coulomb's law of dry friction states that, when two bodies are in contact, the force required to produce sliding is proportional to the normal force acting in the plane of contact. Thus, the friction force *F* is given by:

$$F = \mu N = \mu W = \mu mg \qquad (2.106)$$

where N is normal force,

μ is the coefficient of sliding or kinetic friction

μ is usu 0.1 for lubricated metal, 0.3 for nonlubricated metal on metal, 1.0 for rubber on metal

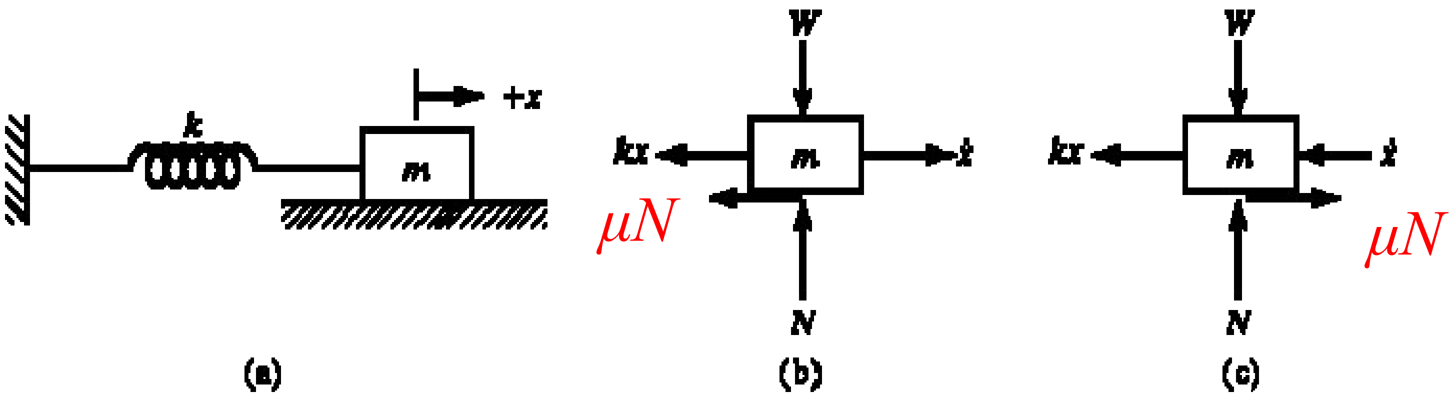
- Coulomb damping is sometimes called *constant damping*

Sec 2.7

Free Vibration with Coulomb Damping

- Equation of Motion:

Consider a single degree of freedom system with dry friction as shown in Fig.(a) below.



Since friction force varies with the direction of velocity, we need to consider two cases as indicated in Fig.(b) and (c).

Free Vibration with Coulomb Damping

Case 1. When x is positive and dx/dt is positive or when x is negative and dx/dt is positive (i.e., for the half cycle during which the mass moves from left to right) the equation of motion can be obtained using Newton's second law (Fig. b):

$$m\ddot{x} = -kx - \mu N \quad \text{or} \quad m\ddot{x} + kx = -\mu N \tag{2.107}$$

Hence,

$$x(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t \boxed{- \frac{\mu N}{k}} \tag{2.108}$$

where $\omega_n = \sqrt{k/m}$ is the frequency of vibration
A_1 & A_2 are constants

Free Vibration with Coulomb Damping

Case 2. When x is positive and dx/dt is negative or when x is negative and dx/dt is negative (i.e., for the half cycle during which the mass moves from right to left) the equation of motion can be derived from Fig. (c):

$$-kx + \mu N = m\ddot{x} \quad \text{or} \quad m\ddot{x} + kx = \mu N \tag{2.109}$$

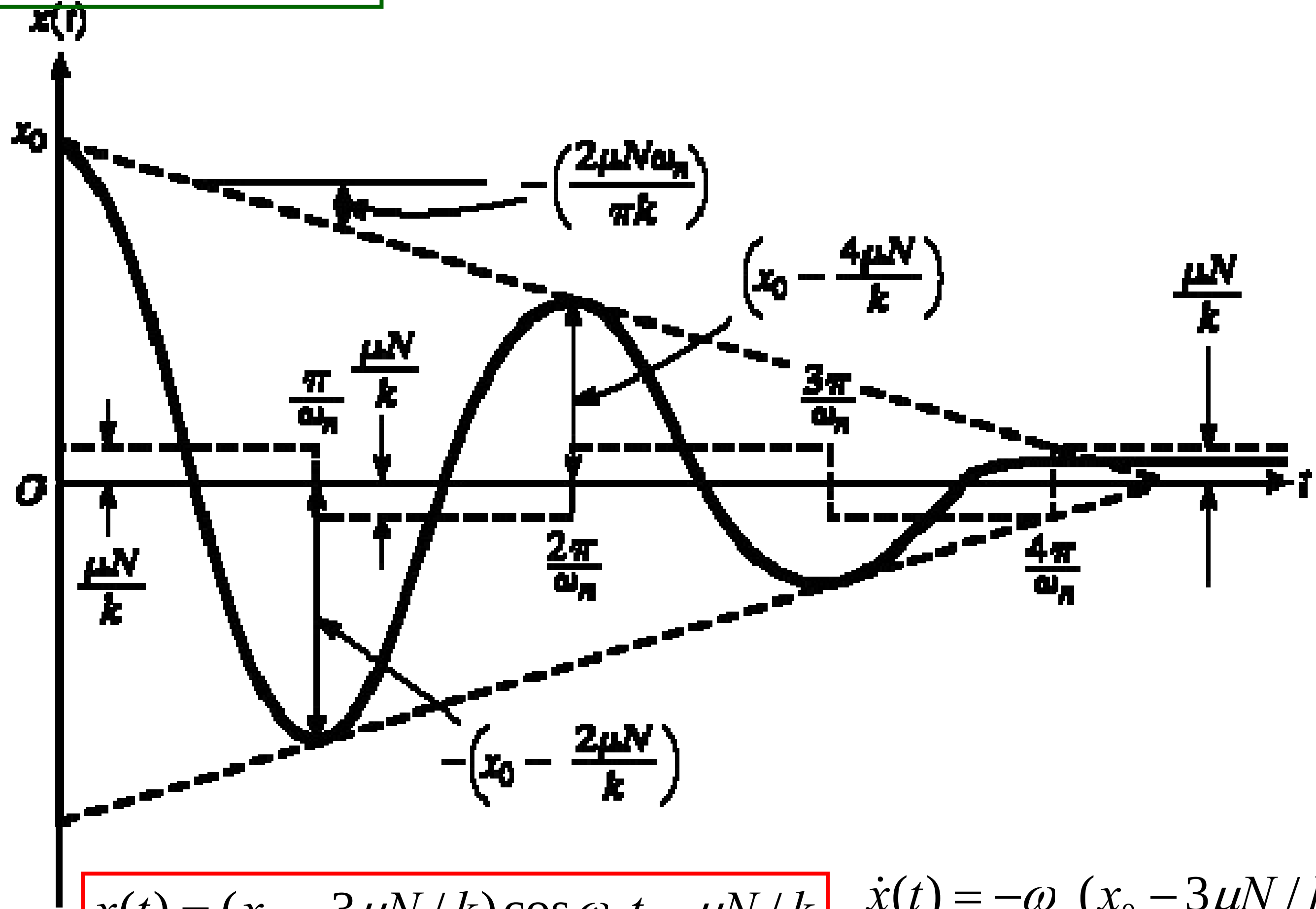
The solution of the equation is given by:

$$x(t) = A_3 \cos\omega_n t + A_4 \sin\omega_n t \boxed{+\frac{\mu N}{k}} \tag{2.110}$$

where A_3 & A_4 are constants

Free Vibration with Coulomb Damping

$$\boxed{x(t) = (x_0 - \mu N / k)\cos\omega_n t + \mu N / k} \qquad \dot{x}(t) = -(x_0 - \mu N / k)\omega_n \sin\omega_n t$$



$$\boxed{x(t) = (x_0 - 3\mu N / k)\cos\omega_n t - \mu N / k} \qquad \dot{x}(t) = -\omega_n (x_0 - 3\mu N / k)\sin\omega_n t$$

Fig. 2.34 Motion of the mass with Coulomb damping

Free Vibration with Coulomb Damping

- Solution:

Eqs.(2.107) & (2.109) can be expressed as a single equation using *N* = *mg:*

$$m\ddot{x} + \mu mg\,\mathrm{sgn}(\dot{x}) + kx = 0 \qquad (2.111)$$

where sgn(y) is called the signum function, whose value is defined as 1 for y > 0, -1 for y< 0, and 0 for y = 0.

Assuming initial conditions as

$$x(t=0) = x_0$$

$$\dot{x}(t=0) = 0 \qquad (2.112)$$

Free Vibration with Coulomb Damping

The solution is valid for half the cycle only, i.e., for $0 \le t \le \pi/\omega_n$. Hence, the solution becomes the initial conditions for the next half cycle. The procedure continued until the motion stops, i.e., when $x_n \le \mu N/k$. Thus the number of half cycles (r) that elapse before the motion ceases is:

$$x_0 - r\frac{2\mu N}{k} \le \frac{\mu N}{k}$$

$$\boxed{\begin{aligned} &A_3 = x_0 - \mu N/k, A_4 = 0 \\ &x(t) = (x_0 - \mu N/k)\cos\omega_n t + \mu N/k \\ &\dot{x}(t) = -(x_0 - \mu N/k)\omega_n \sin\omega_n t \end{aligned}}$$

That is, $r \ge \left\{ \dfrac{x_0 - \dfrac{\mu N}{k}}{\dfrac{2\mu N}{k}} \right\}$ $\quad (2.115)$

$$-A_1 = -x_0 + 3\mu N/k, A_2 = 0$$

$$x(t = \pi/\omega_n) = (x_0 - \mu N/k)\cos\pi + \mu N/k = -(x_0 - 2\mu N/k)$$

$$x(t) = (x_0 - 3\mu N/k)\cos\omega_n t - \mu N/k$$

$\pi/\omega_n \le t \le 2\pi/\omega_n$

Free Vibration with Coulomb Damping

Note the following characteristics of a system with Coulomb damping:

1. The equation of motion is nonlinear with Coulomb damping, while it is linear with viscous damping
2. The natural frequency of the system is unaltered with the addition of Coulomb damping, while it is reduced with the addition of viscous damping.
3. The motion is periodic with Coulomb damping, while it can be nonperiodic in a viscously damped (overdamped) system.
4. The system comes to rest after some time with Coulomb damping, whereas the motion theoretically continues forever (perhaps with an infinitesimally small amplitude) with viscous damping.

Free Vibration with Coulomb Damping

Note the following characteristics of a system with Coulomb damping:

5. The amplitude reduces linearly with Coulomb damping, whereas it reduces exponentially with viscous damping.
6. In each successive cycle, the amplitude of motion is reduced by the amount 4μN/k, so the amplitudes at the end of any two consecutive cycles are related:

$$X_m = X_{m-1} - \frac{4\mu N}{k} \qquad (2.116)$$

As amplitude is reduced by an amount $4\mu N/k$ in one cycle, the slope of the enveloping straight lines (shown dotted) in Fig 2.34.

Free Vibration with Coulomb Damping

- Torsional Systems with Coulomb Damping:

The equation governing the angular oscillations of the system is

$$J_0\ddot{\theta} + k_t\theta = -T \qquad (2.117)$$

and

$$J_0\ddot{\theta} + k_t\theta = T \qquad (2.118)$$

The frequency of vibration is given by

$$\omega_n = \sqrt{\frac{k_t}{J_0}} \qquad (2.119)$$

Free Vibration with Coulomb Damping

and the amplitude of motion at the end of the *r*-th half cycle (θ_r) is given by:

$$\theta_r = \theta_0 - r\frac{2T}{k_t} \qquad (2.120)$$

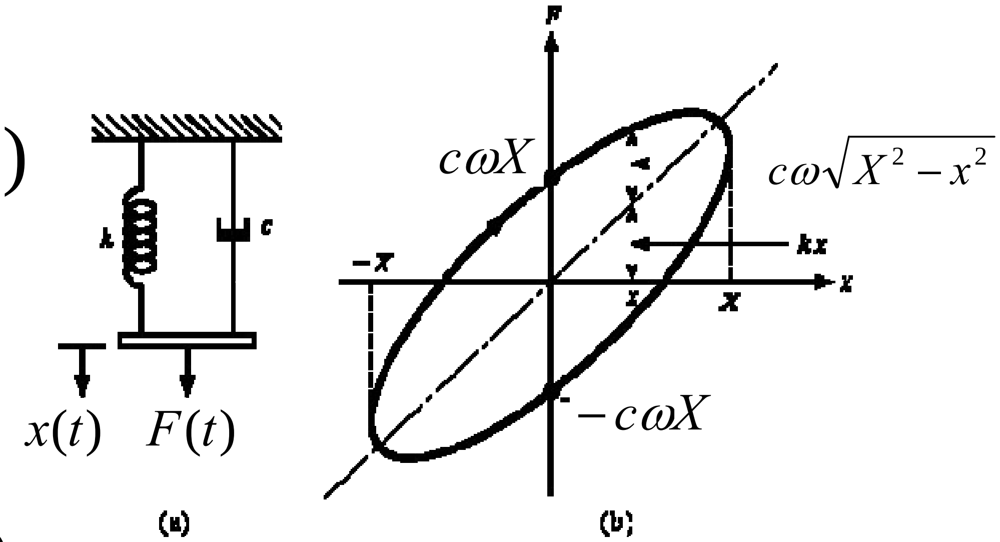
The motion ceases when

$$r \geq \left\{ \frac{\theta_0 - \dfrac{T}{k_t}}{\dfrac{2T}{k_t}} \right\} \qquad (2.121)$$

Free Vibration with Hysteretic Damping

Consider the spring-viscous damper arrangement shown in the figure below. The force needed to cause a displacement:

$$F = kx + c\dot{x} \qquad (2.122)$$



For a harmonic motion of frequency ω and amplitude X,

$$x(t) = X\sin\omega t \quad (2.123)$$

$$\therefore \qquad F(t) = kX\sin\omega t + cX\omega\cos\omega t$$

$$= kx \pm c\omega\sqrt{X^2 - (X\sin\omega t)^2}$$

$$= kx \pm c\omega\sqrt{X^2 - x^2} \qquad (2.124)$$

Free Vibration with Hysteretic Damping

When F versus x is plotted, Eq.(2.124) represents a closed loop, as shown in Fig(b). The area of the loop denotes the energy dissipated by the damper in a cycle of motion and is given by:

$$\Delta W = \oint F dx = \int_0^{2\pi/\omega} (kX \sin \omega t + cX\omega \cos \omega t)(\omega X \cos \omega t) dt$$

$$= \pi \omega c X^2 \qquad (2.125)$$

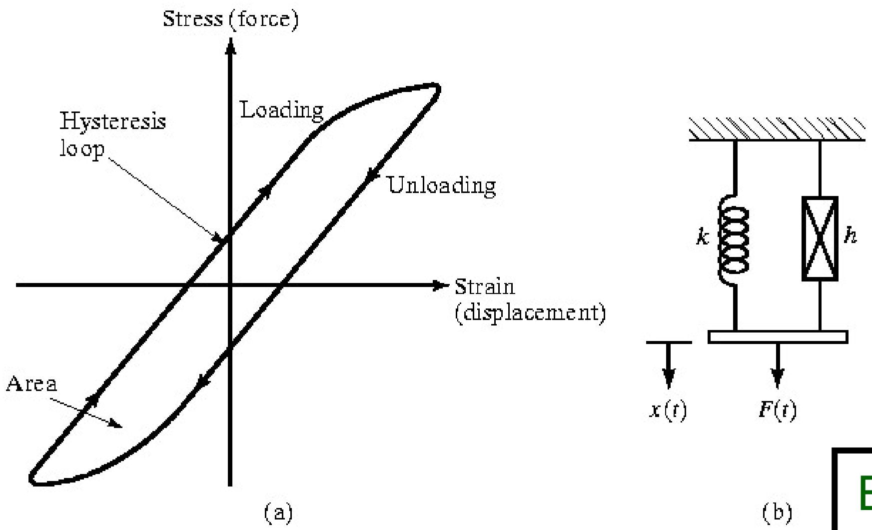


Fig.2.36 Hysteresis loop

Hence, the damping coefficient:

$$c = \frac{h}{\omega} \qquad (2.126)$$

where h is called the hysteresis damping constant.

Experiments shows that the energy loss due to internal friction is independent of operating ω

Free Vibration with Hysteretic Damping

Eqs.(2.125) and (2.126) gives

$$\Delta W = \pi h X^2 \tag{2.127}$$

Complex Stiffness.

For general harmonic motion, $x = Xe^{i\omega t}$, the force is given by

$$F = kXe^{i\omega t} + c\omega i X e^{i\omega t} = (k + i\omega c)x \tag{2.128}$$

Thus, the force-displacement relation:

$$F = \underline{(k + ih)}x \tag{2.129}$$

where $$k + ih = k\left(1 + i\frac{h}{k}\right) = k(1 + i\beta) \tag{2.130}$$

Free Vibration with Hysteretic Damping

Response of the system.

The energy loss per cycle can be expressed as

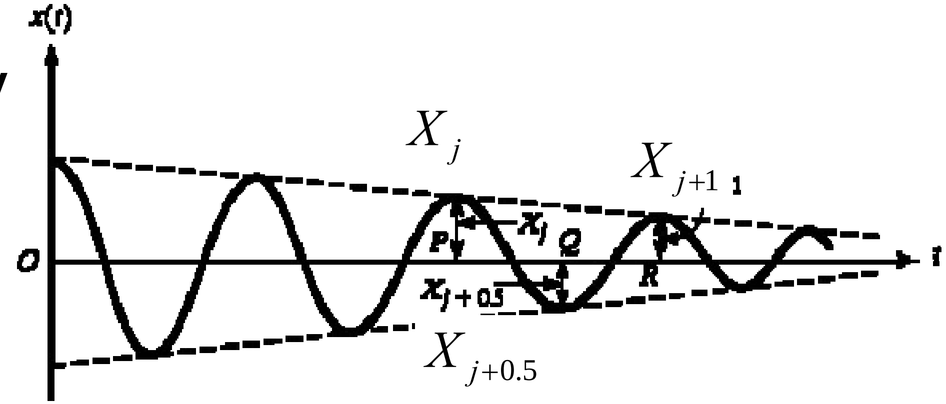
$$\Delta W = \pi k \beta X^2 \qquad (2.131)$$

The hysteresis logarithmic decrement can be defined as

$$\delta = \ln\left(\frac{X_j}{X_{j+1}}\right) \approx \ln(1+\pi\beta) \approx \pi\beta \qquad (2.135)$$

Corresponding frequency

$$\omega = \sqrt{\frac{k}{m}} \qquad (2.136)$$



Response of a hysteretically damped system

Free Vibration with Hysteretic Damping

The equivalent viscous damping ratio

$$\delta \approx 2\pi\zeta_{eq} \approx \pi\beta = \frac{\pi h}{k}$$

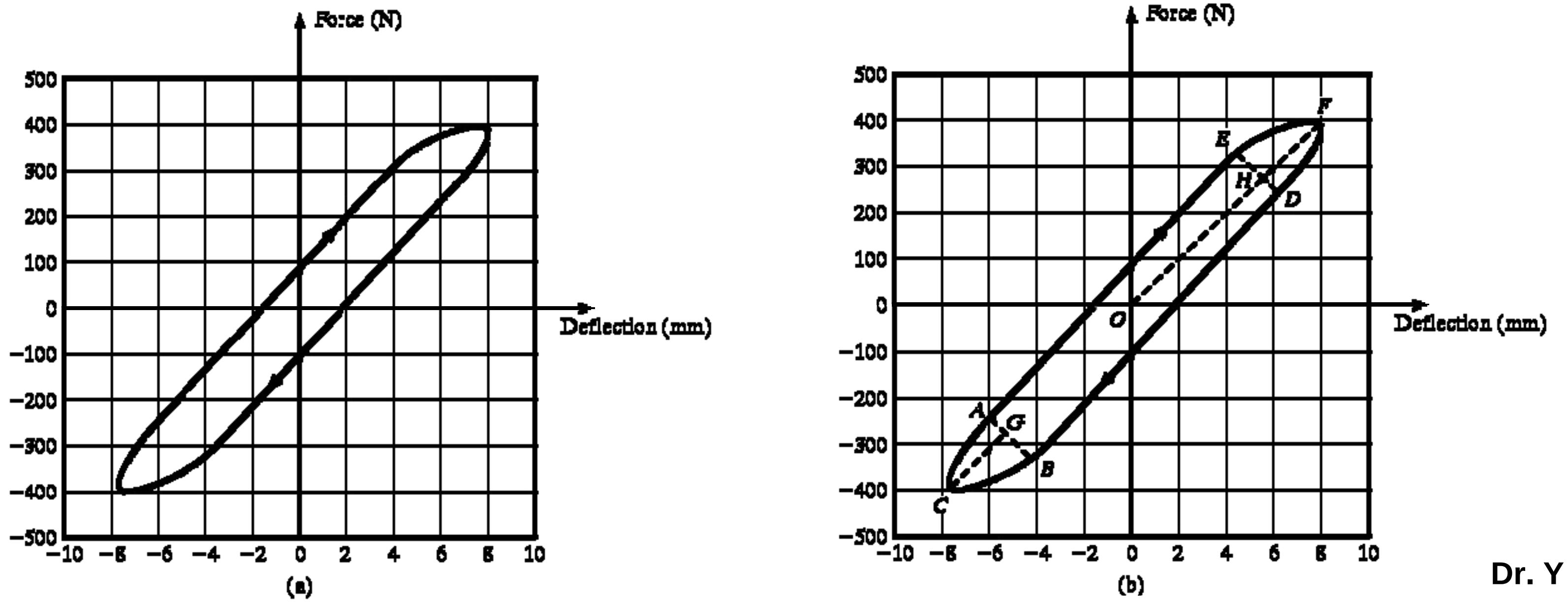
$$\zeta_{eq} = \frac{\beta}{2} = \frac{h}{2k} \qquad (2.137)$$

And thus the equivalent damping constant is

$$c_{eq} = c_c \cdot \zeta_{eq} = 2\sqrt{mk} \cdot \frac{\beta}{2} = \beta\sqrt{mk} = \frac{\beta k}{\omega} = \frac{h}{\omega} \qquad (2.138)$$

Example 2.16 Response of a Hysteretically Damped Bridge Structure

A bridge structure is modeled as a single degree of freedom system with an equivalent mass of 5×10^5 kg and an equivalent stiffness of 25×10^6 N/m. During a free vibration test, the ratio of successive amplitudes was found to be 1.04. Estimate the structural damping constant (β) and the approximate free vibration response of the bridge.



Example 2.16 Solution

Using the ratio of successive amplitudes, Eq.(2.135) yields the hysteresis logarithmic decrement as

$$\delta = \ln\left(\frac{X_j}{X_{j+1}}\right) = \ln(1.04) = \ln(1+\pi\beta)$$

$$1+\pi\beta = 1.04 \quad or \qquad \beta = \frac{0.04}{\pi} = 0.0127$$

The equivalent viscous damping coefficient is

$$c_{eq} = \frac{\beta k}{\omega} = \frac{\beta k}{\sqrt{\frac{k}{m}}} = \beta\sqrt{km} \qquad \text{(E.1)}$$

Example 2.16 Solution

Using the known values of the equivalent stiffness and equivalent mass,

$$c_{eq} = (0.0127)\sqrt{(25\times10^6)(5\times10^5)} = 44.9013\times10^3 \text{ N-s/m}$$

Since $c_{eq} < c_c$, the bridge is underdamped.
Hence, its free vibration response is

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0 \cos\sqrt{1-\zeta^2}\,\omega_n t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n}\sin\sqrt{1-\zeta^2}\,\omega_n t\right\}$$

where $$\zeta = \frac{c_{eq}}{c_c} = \frac{40.9013\times10^3}{7071.0678\times10^3} = 0.0063$$

2.6.4 Energy dissipated in Viscous Damping:

In a viscously damped system, the rate of change of energy with time is given by:

$$\frac{dW}{dt} = \text{force} \times \text{velocity} = Fv = -cv^2 = -c\left(\frac{dx}{dt}\right)^2 \qquad (2.93)$$

The energy dissipated in a complete cycle is:

$$\Delta W = \int_{t=0}^{(2\pi/\omega_d)} c\left(\frac{dx}{dt}\right)^2 dt = \int_0^{2\pi} cX^2\omega_d \cos^2 \omega_d t \cdot d(\omega_d t)$$

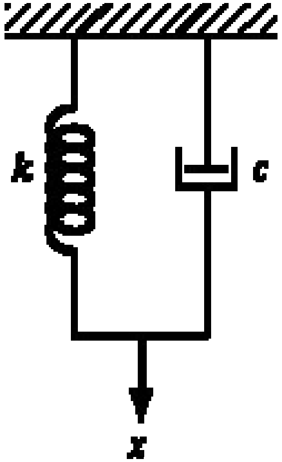
$$= \pi c \omega_d X^2 \qquad (2.94)$$

Energy dissipation

Consider the system shown in the figure below.

The total force resisting the motion is:

$$F = -kx - cv = -kx - c\dot{x} \qquad (2.95)$$



If we assume simple harmonic motion:

$$x(t) = X \sin \omega_d t \qquad (2.96)$$

Thus, Eq.(2.95) becomes

$$F = -kX \sin \omega_d t - c\omega_d X \cos \omega_d t \qquad (2.97)$$

The energy dissipated in a complete cycle will be

$$\Delta W = \int_{t=0}^{2\pi/\omega_d} Fvdt$$

$$= \int_{t=0}^{2\pi/\omega_d} kX^2 \omega_d \sin \omega_d t \cdot \cos \omega_d t \cdot d(\omega_d t)$$

$$+ \int_{t=0}^{2\pi/\omega_d} c\omega_d X^2 \cos^2 \omega_d t \cdot d(\omega_d t) = \pi c \omega_d X^2 \qquad (2.98)$$

Energy dissipation and Loss Coefficient

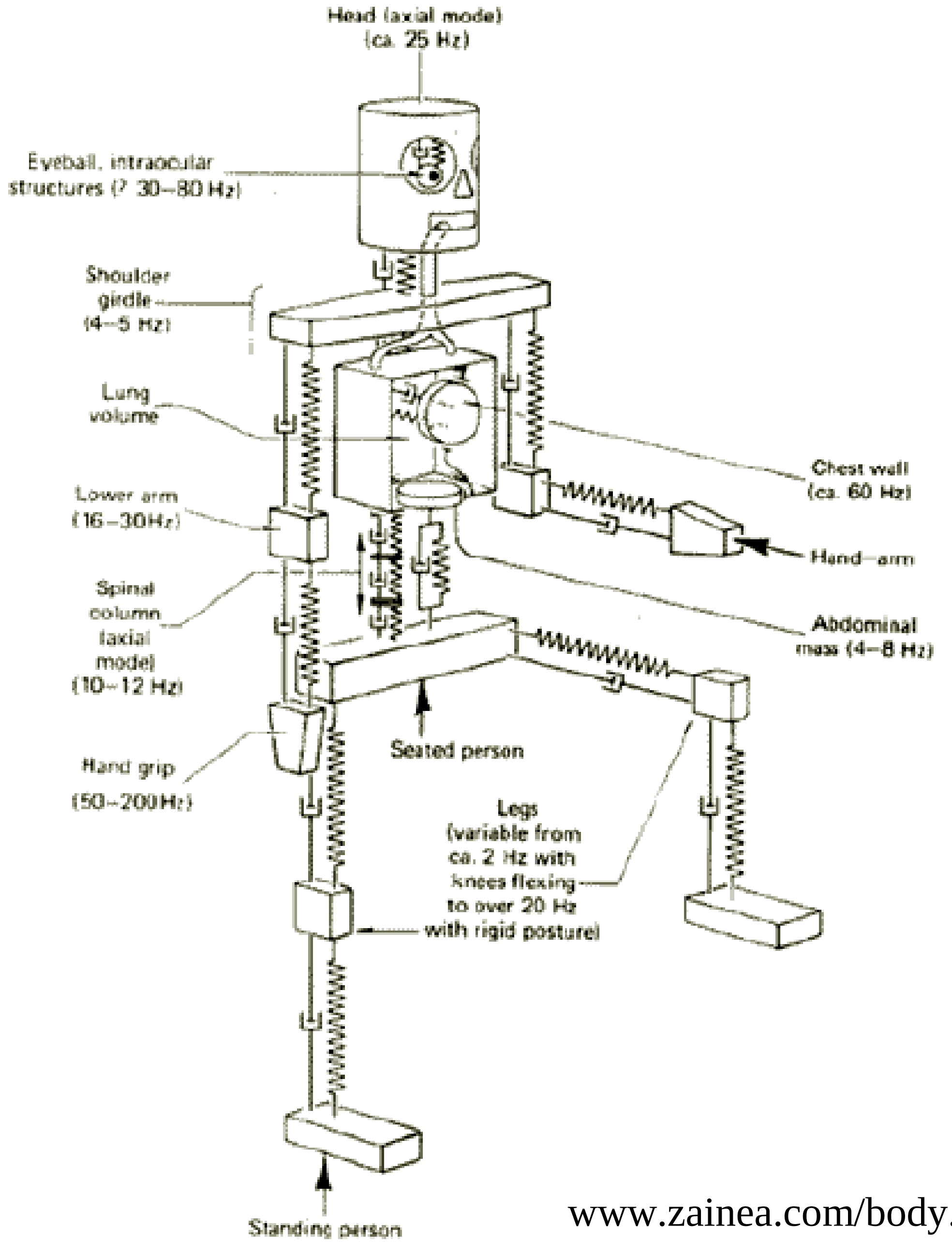
Computing the fraction of the total energy of the vibrating system that is dissipated in each cycle of motion, Specific Damping Capacity

$$\frac{\Delta W}{W} = \frac{\pi c \omega_d X^2}{\frac{1}{2} m \omega_d^2 X^2} = 2\left(\frac{2\pi}{\omega_d}\right)\left(\frac{c}{2m}\right) = 2\delta \approx 4\pi\zeta = \text{constant} \qquad (2.99)$$

where *W* is either the max potential energy or the max kinetic energy.

The loss coefficient, defined as the ratio of the energy dissipated per radian and the total strain energy:

$$\text{loss coefficient} = \frac{(\Delta W / 2\pi)}{W} = \frac{\Delta W}{2\pi W} \qquad (2.100)$$



www.zainea.com/body.htm

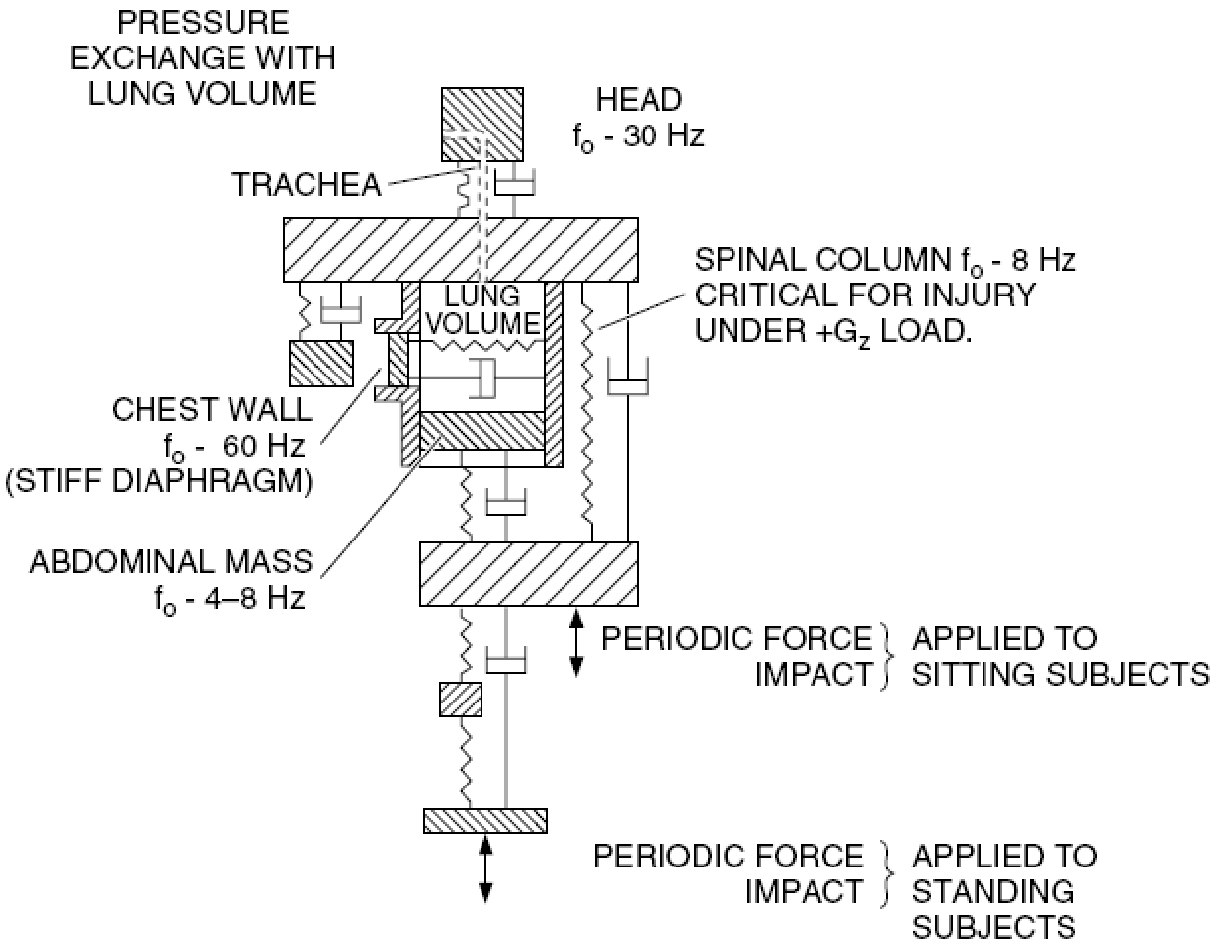


FIGURE 42.5 Lumped parameter biodynamic model of the standing and sitting human body for calculating motion of body parts and some physiological and subjective responses to vertical vibration. The approximate resonance frequencies of various subsystems are indicated by f_o. (*von Gierke.*[6])